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INTRODUCTIO
IN ANALYSIN
INFINITORUM.

AUCTORE

LEONHARDO EULERO,

Profeſſore Regio BEROLINENSI, *& Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

EDITIO NOVA.

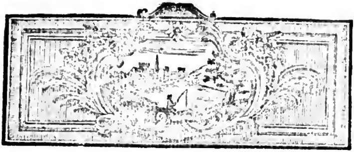
TOMUS PRIMUS.



LUGDUNI,
Apud BERNUSET, DELAMOLLIERE, FALQUE & Soc.

1797.

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ILLUSTRISSIMO VIRO

JOHANNI JACOBO

DORTOUS DE MAIRAN,

UNI EX XL VIRIS

ACADEMIÆ GALLICÆ,

REGIÆ ETIAM SCIENTIARUM

PARISIENSIS,

IN QUA SECRETARII PERPETUI MUNUS NUPER

ABDICAVIT,

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NEC NON

ALIARUM BENE MULTARUM,

LONDINENSIS, PETROPOLITANÆ,

&c.

SOCIETATUM, ACADEMIARUMVE

SOCIO DIGNISSIMO.

MARCUS-MICHAEL BOUSQUET.

VIR ILLUSTRISSIME,

PATRONOS Euleriano *ſcripto quærere neceſſe neutiquam eſſe, Mathematicarum Diſciplinarum cultoribus ſatis conſtat. Sciunt utique illi, varias earum partes novis eum luminibus ſic illuſtraſſe, ut indè meritò clariſſimi rerum in his abſtruſiſſimarum interpretis locum ſit conſequutus. Quem quin egregiè tueatur, immò tollat ſe altiùs quoque opere iſthoc, nemo dubitabit, certior hiſce factus, indulſiſſe* Te *mihi, ut illuſtriſſimo nomini* Tuo *dicatum publicè prodiret. Pertinere autem hoc in me collatum beneficium ad Auctoris decus probe intelligens, Ipſe, ut eo uterer, lubens conceſſit; & cum in rem meam faciat omnimodò, quî neglexiſſem?*

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Ab his equidem, quibus Libros inſcribunt, ſibi neſcio quid ideò deberi, plerique tacitè conſtituunt; accepiaque beneficia quodammodo remunerari, ut ſeſe ferè nexu liberent omni. Ego verò ſecus ſentio. Mihi certè merum eſt beneficium patroni, quòd ſcriptoris aut excuſoris opera id genus honore condecorari patiantur. Hac mente utique Tibi, Vir Illustrissime, *animi gratiſſimi ſummæque obſervantiæ profeſſionem hiſce publicam excipias, rogo.*

Paratum promptumque ſemper juvandis litterarum ſtudiis qui Te *novit, & notus vel hoc nomine es cuicumque in Republica doctorum Europæ totius non hoſpiti, plurimis officiis meæ etiam conditionis homines à* Te *affectos fuiſſe ſtatuat neceſſe eſt. Nempe, tanquam* Tibi *uni eſſet injunctum curare, ut floreant humanum ingenium illuſtrantes ſcientiæ omnes, hominumque in uſus adinventæ artes, ad ſingulis inſervientium artifices etiam* Te *demittere dignaris, vel ab illa ſublimium rerum perſcrutatione, Cælive ipſius* Tibi *tam nota regione, ut quæ hucuſque mentes hominum metu complebant phænomena minus intellecta, per* Te *jam grato tantum admirationis ſenſu contemplentur, earumque cauſas habeant perſpectas.*

Hinc ille veluti ex condicto Academiarum Orbis eruditi concurſus, ut adlectum Te *cætui ſuo conſequerentur, ornamento aliàs carituro inſigni, quo cæteras nollent præ ſe frui. Hinc inprimis Illuſtriſſimæ Pariſienſis de* Te *judicium, cum ageretur*

de ſucceſſore ſufficiendo in locum emeriti Fontenellii , *Viri ; cujus ex ore calamoque fluere Scientiarum Artiumque omnium exquiſitiores divitiæ , elegantiæque univerſæ perpetuo viſæ ſunt , & videbuntur dum ſani ſenſus quicquam humano ingenio erit.* Tibi , *ſcilicet , Commentariorum Academiæ conſcribendorum provincia , cui præfectus ille erat , demandabatur continuo ; quam , ut ornare diutiùs voluiſſes , docti omnes optabant : hoc uno minùs dolentes* Te *aliter cenſuiſſe , quòd aliis* Tibi *magis placituris , profuturiſque nihilominus litteris in univerſum eruditionis ingeniive theſauros impenderes. Quod ut ad ultimas uſque metas hominum vitæ poſitas incolumis , florens , atque beatus præſtes , omni votorum contentione precor. Vale !*

Dabam *Lauſannæ* die 1. Aprilis.
Anni Æræ Dionyſ. 1748.

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PRÆFATIO.

Sæpe numero animadverti, maximam difficultatum partem; quas Matheſeos cultores in addiſcenda Analyſi infinitorum offendere ſolent, inde oriri, quod, Algebra communi vix apprehenſa, animum ad illam ſublimiorem artem appellant; quo fit, ut non ſolum quaſi in limine ſubſiſtant, ſed etiam perverſas ideas illius infiniti, cujus notio in ſubſidium vocatur, ſibi forment. Quanquam autem Analyſis infinitorum non perfectam Algebræ communis, omniumque artificiorum adhuc inventorum cognitionem requirit; tamen plurimæ extant quæſtiones, quarum evolutio diſcentium animos ad ſublimiorem ſcientiam præparare valet, quæ tamen in communibus Algebræ elementis, vel omittuntur, vel non ſatis accurate pertractantur. Hanc ob rem non dubito, quin ea, quæ in his libris congeſſi, hunc defectum abunde ſupplere queant. Non ſolum enim operam dedi, ut eas res, quas Analyſis infinitorum abſolute requirit, uberius atque diſtinctius exponerem, quam vulgo fieri ſolet; ſed etiam ſatis multas quæſtiones enodavi, quibus Lectores ſenſim & quaſi præter expectationem ideam infiniti ſibi familiarem reddent. Plures quoque quæſtiones per præcepta communis Algebræ hic reſolvi, quæ vulgo in Analyſi infini-

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torum tractantur: quo facilius deinceps utriuſque Methodi ſummus conſenſus eluceat.

Diviſi hoc Opus in duos Libros, in quorum priori, quæ ad meram Analyſin pertinent, ſum complexus: in poſteriori vero, quæ ex Geometria ſunt ſcitu neceſſaria, explicavi, quoniam Analyſis infinitorum ita quoque tradi ſolet, ut ſimul ejus applicatio ad Geometriam oſtendatur. In utroque autem prima Elementa prætermiſi, eaque tantum exponenda duxi, quæ alibi, vel omnino non, vel minus commode tractata, vel ex diverſis principiis petita reperiuntur.

In primo igitur Libro, cum univerſa Analyſis infinitorum circa quantitates variabiles earumque Functiones verſetur, hoc argumentum de Functionibus imprimis fuſius expoſui; atque Functionum tam transformationem, quam reſolutionem & evolutionem per ſeries infinitas demonſtravi. Complures enumeravi Functionum ſpecies, quarum in Analyſi ſublimiori præcipue ratio eſt habenda. Primum eas diſtinxi in algebraicas & tranſcendentes; quarum illæ per operationes in Algebra communi uſitatas ex quantitatibus variabilibus formantur, hæ vero vel per alias rationes componuntur, vel ex iiſdem operationibus infinities repetitis efficiuntur. Algebraicarum functionum primaria ſubdiviſio fit in rationales & irrationales, priores docui cum in partes ſimpliciores, tum in factores reſolvere; quæ operatio in Calculo integrali maximum adjumentum affert; poſteriores vero, quemadmodum idoneis ſubſtitutionibus ad formam rationalem perduci queant oſtendi. Evolutio autem per ſeries infinitas ad utrumque genus æque pertinet, atque etiam ad Functiones tranſcendentes ſumma cum utilitate applicari ſolet; atquantopere doctrina de ſeriebus infinitis Analyſin ſublimiorem amplificaverit, nemo eſt qui ignoret. Nonnulla igitur adjunxi Capita, quibus plurium ſerierum infinitarum proprietates, atque ſummas ſum ſcrutatus; quarum quædam ita ſunt comparatæ, ut ſine ſubſidio Analyſis infinitorum vix inveſtigari poſſe videantur. Hujuſmodi

ſeries ſunt, quarum ſummæ exprimuntur, vel per Logarithmos vel Arcus circulares: quæ quantitates cum ſint tranſcendentes, dum per quadraturam Hyperbolæ & Circuli exhibentur, maximam partem demum in Analyſi infinitorum tractari ſunt ſolitæ. Poſtquam autem a poteſtatibus ad quantitates exponentiales eſſem progreſſus, quæ nil aliud ſunt niſi poteſtates, quarum exponentes ſunt variabiles; ex earum converſione maxime naturalem ac fœcundam Logarithmorum ideam ſum adeptus: unde non ſolum ampliſſimus eorum uſus ſponte eſt conſecutus, ſed etiam ex ea cunctas ſeries infinitas, quibus vulgo iſtæ quantitates repræſentari ſolent, elicere licuit: hincque adeo facillimus ſe prodidit modus Tabulas Logarithmorum conſtruendi. Simili modo in contemplatione Arcuum circularium ſum verſatus; quod quantitatum genus, etſi a Logarithmis maxime eſt diverſum, tamen tam arcto vinculo eſt connexum, ut dum alterum imaginarium fieri videtur, in alterum tranſeat. Repetitis autem ex Geometria quæ de inventione Sinuum & Coſinuum Arcuum multiplorum ac ſubmultiplorum traduntur, ex Sinu vel Coſinu cujusque Arcus expreſſi Sinum Coſinumque Arcus minimi & quaſi evaneſcentis, quo ipſo ad ſeries infinitas ſum deductus: unde, cum Arcus evaneſcens Sinui ſuo ſit æqualis, Coſinus vero radio, quemvis Arcum cum ſuo Sinu & Coſinu ope ſerierum infinitarum comparavi. Tum vero tam varias expreſſiones cum finitas tum infinitas pro hujus generis quantitatibus obtinui, ut ad earum naturam perſpiciendam Calculo infiniteſimali prorſus non amplius eſſet opus. Atque quemadmodum Logarithmi peculiarem Algorithmum requirunt, cujus in univerſa Analyſi ſummus extat uſus, ita quantitates circulares ad certam quoque Algorithmi normam perduxi; ut in calculo æque commode ac Logarithmi & ipſæ quantitates algebraicæ tractari poſſent. Quantum autem hinc utilitatis ad reſolutionem difficillimarum quæſtionum redundet, cum nonnulla Capita hujus Libri luculenter declarant, tum ex Analyſi infinitorum plurima ſpeci-

mina proferri poſſent, niſi jam ſatis eſſent cognita & in dies magis multiplicarentur. Maximum autem hæc inveſtigatio attulit adjumentum ad Functiones fractas in factores reales reſolvendas; quod argumentum, cum in Calculo integrali ſit prorſus neceſſarium, diligentius enucleavi. Series poſtmodum infinitas, quæ ex hujusmodi Functionum evolutione naſcuntur & quæ recurrentium nomine innotuerunt, examini ſubjeci; ubi earum tam ſummas quam terminos generales, aliaſque inſignes proprietates exhibui: & quoniam ad hæc reſolutio in factores manuduxit, ita viciſſim, quemadmodum producta ex pluribus, imo etiam infinitis, factoribus conſlata per multiplicationem in ſeries explicentur, perpendi. Quod negotium non ſolum ad cognitionem innumerabilium ſerierum viam aperuit, ſed quia hoc modo ſeries in producta ex infinitis factoribus conſtantia reſolvere licebat, ſatis commodas inveni expreſſiones numericas, quarum ope Logarithmi Sinuum, Coſinuum & Tangentium facillime ſupputari poſſunt. Præterea quoque ex eodem fonte ſolutiones plurium quæſtionum, quæ circa partitionem numerorum proponi poſſunt, derivavi; cujuſmodi quæſtiones ſine hoc ſubſidio vires Analyſeos ſuperare videantur. Hæc tanta materiarum diverſitas in plura volumina facile excreſcere potuiſſet; ſed omnia, quantum fieri potuit, tam ſuccincte propoſui, ut ubique fundamentum clariſſime quidem explicaretur, uberior vero amplificatio induſtriæ Lectorum relinqueretur; quo habeant, quibus vires ſuas exerceant, fineſque Analyſeos ulterius promoveant. Neque enim vereor profiteri, in hoc Libro non ſolum multa plane nova contineri; ſed etiam fontes eſſe detectos, unde plurima inſignia inventa adhuc hauriri queant. Eodem inſtituto ſum uſus in altero Libro, ubi, quæ vulgo ad Geometriam ſublimiorem referri ſolent, pertractavi. Antequam autem de Sectionibus Conicis, quæ alias fere ſolæ hunc locum occupant, agerem; Theoriam Linearum Curvarum in genere ita propoſui, ut ad ſcrutationem naturæ

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quarumvis Linearum Curvarum cum utilitate adhiberi poſſet. Ad hoc nullum aliud ſubſidium affero, præter æquationem, qua cujuſque Lineæ Curvæ natura exprimitur; ex eaque cum figuram, tum primarias proprietates deducere doceo: id quod potiſſimum in Sectionibus Conicis præſtitiſſe mihi ſum viſus; quæ antehac vel ſecundum ſolam Geometriam vel per Analyſin quidem, ſed nimis imperfecte ac minus naturaliter, tractari ſunt ſolitæ. Ex æquatione ſcilicet generali pro Lineis ſecundi ordinis primum earum proprietates generales explicavi, tum eas in genera ſeu ſpecies ſubdiviſi; reſpiciendo utrum habeant ramos in infinitum excurrentes, an vero tota Curva finito ſpatio includatur. Priori autem caſu inſuper diſpiciendum erat, quot ſint rami in infinitum excurrentes, & cujus naturæ ſint ſinguli; an habeant Lineas rectas aſymtotas, an minus. Sicque obtinui tres conſuetas Sectionum Conicarum ſpecies; quarum prima eſt Ellipſis, tota in ſpatio finito contenta; ſecunda autem Hyperbola, quæ quatuor habet ramos infinitos ad duas rectas aſymptotas convergentes; tertia vero ſpecies prodiit Parabola duos habens ramos infinitos aſymtotis deſtitutos. Simili porro ratione Lineas tertii ordinis ſum perſecutus, quas, poſt expoſitas earum proprietates generales, diviſi in ſedecim genera; ad eaque omnes ſeptuagenta duas ſpecies Newtoni revocavi. Ipſam vero methodum ita clare deſcripſi, ut pro quovis Linearum ordine ſequente diviſio in genera facillime inſtitui queat; cujus negotii periculum quoque feci in Lineis quarti ordinis. His deinde, quæ ad ordines Linearum pertinent, expeditis, reverſus ſum ad generales omnium Linearum affectiones eruendas. Explicavi itaque methodum definiendi tangentes curvarum, earum normales, atque etiam ipſam curvaturam, quæ per radium oſculi eſtimari ſolet: quæ etſi nunc quidem plerumque Calculo differentiali abſolvuntur, tamen idem per ſolam communem Algebram hic

præſtiti, ut deinceps tranſitus ab Analyſi finitorum ad Analyſin infinitorum eo facilior reddatur. Perpendi etiam curvarum puncta flexus contrarii, cuſpides, puncta duplicia, ac multiplicia; modumque expoſui hæc omnia ex æquationibus ſine ulla difficultate definiendi. Interim tamen non nego, has quæſtiones multo facilius Calculi differentialis ope enodari poſſe. Attigi quoque controverſiam de cuſpide ſecundi ordinis, ubi ambo arcus in cuſpidem coeuntes curvaturam in eandem partem vertunt; eamque ita compoſuiſſe mihi videor, ut nullum dubium amplius ſupereſſe poſſit. Denique adjunxi aliquot Capita, in quibus Lineas Curvas, quæ datis proprietatibus gaudeant, invenire docui; pluraque tandem Problemata circa ſingulares Circuli ſectiones ſoluta dedi. Quæ cum ſint ea ex Geometria, quæ ad Analyſin infinitorum addiſcendam maximum adminiculum afferre videntur, Appendicis loco ex Stereometria Theoriam ſolidorum eorumque ſuperficierum per Calculum propoſui & quemadmodum cujuſque ſuperficiei natura per æquationem inter tres variabiles exponi queat, oſtendi. Hinc, ſuperficiebus inſtar linearum in ordines digeſtis, ſecundum dimenſionum quas variabiles in æquatione conſtituunt numerum, in primo ordine ſolam ſuperficiem planam contineri oſtendi. Superficies vero ſecundi ordinis, ratione habita partium in infinitum expanſarum, in ſex genera diviſi; ſimilique modo pro ceteris ordinibus diviſio inſtitui poterit. Contemplatus ſum quoque interſectiones duarum ſuperficierum; quæ cum plerumque ſint curvæ non in eodem plano ſitæ, quemadmodum æquationibus comprehendi queant, monſtravi. Tandem etiam poſitionem planorum tangentium, atque rectarum, quæ ad ſuperficies ſint normales, determinavi.

De cetero, cum non paucæ res hic occurrant ab aliis jam tractatæ, veniam rogare me oportet, quod non ubique honorificam mentionem eorum, qui ante me in eodem

genere elaborarunt, fecerim. Cum enim mihi propoſitum eſſet omnia quam breviſſime pertractare, Hiſtoria cujuſque Problematis magnitudinem operis non mediocriter auxiſſet. Interim tamen pleræque queſtiones, quæ alibi quoque ſolutæ reperiuntur, hic ſolutiones ex aliis principiis ſunt nactæ; ita ut non exiguam partem mihi vindicare poſſem. Spero autem cum iſta, tum ea potiſſimum, quæ prorſus nova hic proferuntur, pleriſque, qui hoc ſtudio delectantur, non ingrata eſſe futura.

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INTRO-

INTRODUCTIO

IN

ANALYSIN INFINITORUM.

LIBER PRIMUS,

CONTINENS

Explicationem de Functionibus quantitatum variabilium ; earum reſolutione in Factores , atque evolutione per Series infinitas : una cum doctrina de Logarithmis , Arcubus circularibus , eorumque Sinubus & Tangentibus ; pluribuſque aliis rebus , quibus Analyſis infinitorum non mediocriter adjuvatur.

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LIBER PRIMUS.

CAPUT PRIMUM.

DE FUNCTIONIBUS IN GENERE.

Quantitas conſtans eſt quantitas determinata, perpetuo eumdem valorem ſervans.

Ejuſmodi quantitates ſunt numeri cujuſvis generis, quippe qui eumdem, quem ſemel obtinuerunt, valorem conſtanter conſervant: atque ſi hujuſmodi quantitates conſtantes per characteres indicare convenit, adhibentur litteræ Alphabethi initiales *a*, *b*, *c*, &c. In Analyſi quidem communi, ubi tantum quantitates determinatæ conſiderantur, hæ litteræ Alphabethi priores quantitates cognitas denotare ſolent, poſteriores vero quantitates incognitas; at in Analyſi ſublimiori hoc diſcrimen non tantopere ſpectatur, cum hic ad illud quantitatum diſcrimen præcipue reſpiciatur, quo aliæ conſtantes, aliæ vero variabiles ſtatuuntur.

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LIB. I. 2. *Quantitas variabilis eſt quantitas indeterminata ſeu univerſalis, quæ omnes omnino valores determinatos in ſe complectitur.*

Cum ergo omnes valores determinati numeris exprimi queant, quantitas variabilis omnes numeros cujuſvis generis involvit. Quemadmodum ſcilicet ex ideis individuorum formantur ideæ ſpecierum & generum; ita quantitas variabilis eſt genus, ſub quo omnes quantitates determinatæ continentur. Hujuſmodi autem quantitates variabiles per litteras Alphabethi poſtremas z, y, x, &c. repræſentari ſolent.

3. *Quantitas variabilis determinatur, dum ei valor quicunque determinatus tribuitur.*

Quantitas ergo variabilis innumerabilibus modis determinari poteſt, cum omnes omnino numeros ejus loco ſubſtituere liceat. Neque ſignificatus quantitatis variabilis exhauritur, niſi omnes valores determinati ejus loco fuerint ſubſtituti. Quantitas ergo variabilis in ſe complectitur omnes prorſus numeros, tam affirmativos quam negativos, tam integros quam fractos, tam rationales quam irrationales & tranſcendentes. Quinetiam cyphra & numeri imaginarii a ſignificatu quantitatis variabilis non excluduntur.

4. *Functio quantitatis variabilis, eſt expreſſio analytica quomodocunque compoſita ex illa quantitate variabili, & numeris ſeu quantitatibus conſtantibus.*

Omnis ergo expreſſio analytica, in qua præter quantitatem variabilem z omnes quantitates illam expreſſionem componentes ſunt conſtantes, erit Functio ipſius z: Sic $a + 3z$; $az - 4zz$; $az + b\sqrt{(aa - zz)}$; c^z; &c. ſunt Functiones ipſius z.

5. *Functio ergo quantitatis variabilis ipſa erit quantitas variabilis.*

Cum enim loco quantitatis variabilis omnes valores determinatos ſubſtituere liceat, hinc Functio innumerabiles valores determinatos induet; neque ullus valor determinatus excipietur, quem Functio induere nequeat, cum quantitas variabilis quoque valores imaginarios involvat. Sic etſi hæc Functio $\sqrt{(9 - zz)}$, numeris realibus loco z ſubſtituendis, nunquam valorem terna-

rio majorem recipere poteſt ; tamen ipſi z valores imaginarios tribuendo ut $5\sqrt{-1}$, nullus aſſignari poterit valor determinatus quin ex formula $\sqrt{(9-zz)}$ elici queat. Occurrunt autem nonnunquam Functiones tantum apparentes, quæ, utcunque quantitas variabilis varietur, tamen uſque eumdem valorem retinent, ut z^0 ; 1^z ; $\frac{aa-az}{a-z}$, quæ, etſi ſpeciem Functionis mentiuntur, tamen revera ſunt quantitates conſtantes.

6. *Præcipuum Functionum diſcrimen in modo compoſitionis, quo ex quantitate variabili & quantitatibus conſtantibus formantur, poſitum eſt.*

Pendet ergo ab Operationibus quibus quantitates inter ſe componi & permiſceri poſſunt : quæ Operationes ſunt Additio & Subtractio ; Multiplicatio & Diviſio · Evectio ad Poteſtates & Radicum Extractio ; quo etiam Reſolutio Æquationum eſt referenda. Præter has Operationes, quæ algebraïcæ vocari ſolent, dantur complures aliæ tranſcendentes, ut Exponentiales, Logarithmicæ, atque innumerabiles aliæ, quas Calculus integralis ſuppeditat.

Interim ſpecies quædam Functionum notari poſſunt ; ut multipla $2z$; $3z$; $\frac{3}{5}z$; az ; &c. & Poteſtates ipſius z, ut z^2 ; z^3 ; $z^{\frac{1}{2}}$; z^{-1} ; &c. quæ, uti ex unica operatione ſunt deſumtæ, ita expreſſiones quæ ex operationibus quibuſcunque naſcuntur, Functionum nomine inſigniuntur.

7. *Functiones dividuntur in Algebraïcas & Tranſcendentes : illæ ſunt, quæ componuntur per operationes algebraïcas ſolas ; hæ vero, in quibus operationes tranſcendentes inſunt.*

Sunt ergo multiplæ ac Poteſtates ipſius z Functiones algebraïcæ ; atque omnes omnino expreſſiones, quæ per operationes algebraïcas ante memoratas formantur, cujuſmodi eſt $\frac{a+bz^n-c\sqrt{(2z-zz)}}{aaz-3bz^3}$. Quinetiam Functiones algebraïcæ ſæpenumero nequidem explicite exhiberi poſſunt, cujuſmodi Functio ipſius z eſt Z, ſi definiatur per hujuſmodi æquationem ; $Z^5 = azzZ^3 - bz^4 Z^2 + cz^3 Z - 1$. Quanquam enim hæc

 æquatio reſolvi nequit; tamen conſtat Z æquari expreſſioni cuipiam ex variabili z & conſtantibus compoſitæ; ac propterea fore Z Functionem quamdam ipſius z. Cæterum de Functionibus tranſcendentibus notandum eſt, eas demum fore tranſcendentes, ſi operatio tranſcendens non ſolum ingrediatur, ſed etiam quantitatem variabilem afficiat. Si enim operationes tranſcendentes tantum ad quantitates conſtantes pertineant, Functio nihilominus algebraïca eſt cenſenda: uti ſi c denotet circumferentiam Circuli, cujus radius ſit $= 1$, erit utique c quantitas tranſcendens, verumtamen hæ expreſſiones $c + z$; cz^2; $4z^c$ &c. erunt Functiones algebraïcæ ipſius z. Parvi quidem eſt momenti dubium quod a quibuſdam movetur, utrum ejuſmodi expreſſiones z^c Functionibus algebraïcis annumerari jure poſſint, necne; quinetiam Poteſtates ipſius z, quarum exponentes ſint numeri irrationales, uti $z^{\sqrt{2}}$, nonnulli maluerunt Functiones interſcendentes quam algebraïcas appellare.

8. *Functiones algebraïcæ ſubdividuntur in Rationales & Irrationales: illæ ſunt, ſi quantitas variabilis in nulla irrationalitate involvitur; hæ vero, in quibus ſigna radicalia quantitatem variabilem afficiunt.*

In Functionibus ergo rationalibus aliæ operationes præter Additionem, Subtractionem, Multiplicationem, Diviſionem, & Evectionem ad Poteſtates, quarum exponentes ſint numeri integri, non inſunt: erunt adeo $a + z$; $a - z$; az; $\frac{aa + zz}{a + z}$; $az^3 - bz^4$; &c. Functiones rationales ipſius z. At hujuſmodi expreſſiones $\sqrt{z}$; $a + \sqrt{(aa - zz)}$; $\sqrt[3]{(a - 2z + zz)}$: $\frac{aa - z\sqrt{(aa + zz)}}{a + z}$ erunt Functiones irrationales ipſius z.

Hæ commode diſtinguntur in Explicitas & Implicitas.

Explicitæ ſunt, quæ per ſigna radicalia ſunt evolutæ, cujuſmodi exempla modo ſunt data. Implicitæ vero Functiones irrationales ſunt quæ ex reſolutione æquationum ortum habent. Sic Z erit Functio irrationalis implicita ipſius z, ſi per hujuſmodi æquationem $Z^5 = azZ^2 - bz^4$ definiatur; quoniam valorem

explicitum pro Z, admiſſis etiam ſignis radicalibus, exhibere non licet; propterea quod Algebra communis nondum ad hunc perfectionis gradum eſt evecta.

9. *Functiones rationales denuo ſubdividuntur in Integras & Fractas.*

In illis neque z uſquam habet exponentes negativos, neque expreſſiones continent fractiones, in quarum denominatores quantitas variabilis z ingrediatur: unde intelligitur Functiones fractas eſſe, in quibus denominatores z continentes, vel exponentes negativi ipſius z occurrant. Functionum integrarum hæc ergo erit Formula generalis: $a + bz + cz^2 + dz^3 + ez^4 + fz^5 +$ &c. nulla enim Functio ipſius z integra excogitari poteſt, quæ non in hac expreſſione contineatur. Functiones autem fractæ omnes, quia plures fractiones in unam cogi poſſunt, continebuntur in hac Formula:

$$\frac{a+bz+cz^2+dz^3+ez^4+fz^5+\&c.}{\alpha+\beta z+\gamma z^2+\delta z^3+\epsilon z^4+\zeta z^5+\&c.}$$

ubi notandum eſt quantitates conſtantes a, b, c, d, &c. α, β, γ, δ, &c. ſive ſint affirmativæ, ſive negativæ, ſive integræ ſive fractæ, rationales ſive irrationales, ſive etiam tranſcendentes, naturam Functionum non mutare.

10. *Deinde potiſſimum tenenda eſt Functionum diviſio in Uniformes ac Multiformes.*

Functio autem uniformis eſt, quæ ſi quantitati variabili z valor determinatus quicunque tribuatur, ipſa quoque unicum valorem determinatum obtineat. Functio autem Multiformis eſt, quæ, pro unoquoque valore determinato in locum variabilis z ſubſtituto, plures valores determinatos exhibet. Sunt igitur omnes Functiones rationales, ſive integræ ſive fractæ, Functiones uniformes; quoniam ejuſmodi expreſſiones, quicunque valor quantitati variabili tribuatur, non niſi unicum valorem præbent. Functiones autem irrationales omnes ſunt multiformes; propterea quod ſigna radicalia ſunt ambigua, & geminum valorem involvunt. Dantur autem quoque inter Functiones tranſcendentes, & uniformes, & multiformes: quinetiam habentur Func-

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tiones infinitiformes; cujuſmodi eſt Arcus Circuli Sinui z reſpondens; dantur enim Arcus circulares innumerabiles qui omnes eumdem habeant Sinum. Denotent autem hæ litteræ P, Q, R, S, T &c. ſingulæ Functiones uniformes ipſius z.

11. *Functio biformis ipſius z eſt ejuſmodi Functio, quæ pro quovis ipſius z valore determinato, geminum valorem præbeat.*

Hujuſmodi Functiones radices quadratæ exhibent, ut $\sqrt{(2z + zz)}$: quicunque enim valor pro z ſtatuatur expreſſio $\sqrt{(2z + zz)}$ duplicem habet ſignificatum, vel affirmativum vel negativum. Generatim vero Z erit Functio biformis ipſius z, ſi determinetur per æquationem quadraticam $Z^2 - PZ + Q = 0$: ſi quidem P & Q fuerint Functiones uniformes ipſius z. Erit namque $Z = \frac{1}{2} P \pm \sqrt{(\frac{1}{4} P^2 - Q)}$; ex quo patet cuique valori determinato ipſius z duplicem valorem determinatum ipſius Z reſpondere. Hic autem notandum eſt, vel utrumque valorem Functionis Z eſſe realem, vel utrumque imaginarium. Tum vero erit ſemper, uti conſtat ex natura æquationum, binorum valorum ipſius Z ſumma $= P$, ac productum $= Q$.

12. *Functio triformis ipſius z eſt, quæ pro quovis ipſius z valore, tres valores determinatos exhibet.*

Hujuſmodi Functiones ex reſolutione æquationum cubicarum originem trahunt. Si enim fuerint P, Q, & R Functiones uniformes, ſitque $Z^3 - PZ^2 + QZ - R = 0$, erit Z Functio triformis ipſius z; quia pro quolibet valore determinato ipſius z triplicem valorem obtinet. Tres iſti ipſius Z valores unicuique valori ipſius z reſpondentes, vel erunt omnes reales, vel unicus erit realis, dum bini reliqui ſunt imaginarii. Cæterum conſtat horum trium valorum ſummam perpetuo eſſe $= P$; ſummam factorum ex binis eſſe $= Q$, & productum ex omnibus tribus eſſe $= R$.

13. *Functio quadriformis ipſius z eſt, quæ pro quovis ipſius z valore quatuor valores determinatos exhibet.*

Hujuſmodi Functiones ex reſolutione æquationum biquadraticarum

ticarum naſcuntur. Quod ſi enim P, Q, R, & S denotent Functiones uniformes ipſius z, fueritque $Z^4 - PZ^3 + QZ^2 - RZ + S = 0$, erit Z Functio quadriformis ipſius z; eo quod cuique valori ipſius z quadruplex valor ipſius Z reſpondet. Quatuor horum valorum ergo, vel omnes erunt reales, vel duo reales duoque imaginarii, vel omnes quatuor erunt imaginarii. Ceterum perpetuo ſumma horum quatuor valorum ipſius Z eſt $= P$, ſumma factorum ex binis $= Q$, ſumma factorum ex ternis $= R$, ac productum omnium $= S$. Simili autem modo comparata eſt ratio Functionum quinqueformium & ſequentium.

14. *Erit ergo* Z *Functio multiformis ipſius* z, *quæ, pro quovis valore ipſius* z, *tot exhibet valores quot numerus* n *continet unitates; ſi* Z *definiatur per hanc æquationem* $Z^n - PZ^{n-1} + QZ^{n-2} - RZ^{n-3} + SZ^{n-4} - \&c. = 0$.

Ubi quidem notandum eſt n eſſe oportere numerum integrum; atque perpetuo, ut dijudicari poſſit quam multiformis ſit Functio Z ipſius z, æquatio, per quam Z definitur, reduci debet ad rationalitatem; quo facto exponens maximæ poteſtatis ipſius Z indicabit quæſitum valorum numerum cuique ipſius z valori reſpondentium. Deinde quoque tenendum eſt litteras P, Q, R, S, &c. denotare debere Functiones uniformes ipſius z: ſi enim aliqua earum jam eſſet Functio multiformis, tum Functio Z multo plures præbitura eſſet valores unicuique valori ipſius z reſpondentes, quam quidem numerus dimenſionum ipſius Z indicaret. Semper autem, ſi qui valores ipſius Z fuerint imaginarii, eorum numerus erit par; unde intelligitur, ſi fuerit n numerus impar, perpetuo unum ad minimum valorem ipſius Z fore realem: contra autem fieri poſſe, ſi numerus n fuerit par, ut nullus prorſus valor ipſius Z ſit realis.

15. *Si* Z *ejuſmodi fuerit Functio multiformis ipſius* z *ut perpetuo nonniſi unicum valorem exhibeat realem; tum* Z *Functionem uniformem ipſius* z *mentietur, ac plerumque loco Functionis uniformis uſurpari poterit.*

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LIB. I. Ejuſmodi Functiones erunt $\sqrt[3]{P}$, $\sqrt[5]{P}$, $\sqrt[7]{P}$, &c. quippe quæ perpetuo nonniſi unicum valorem realem præbent, reliquis omnibus exiſtentibus imaginariis, dummodo P fuerit Functio uniformis ipſius z. Hanc ob rem hujuſmodi expreſſio $P^{\frac{m}{n}}$, quoties n fuerit numerus impar, Functionibus uniformibus annumerari poterit; ſive m fuerit numerus par ſive impar. Quod ſi autem n fuerit numerus par, tum $P^{\frac{m}{n}}$ vel nullum habebit valorem realem, vel duos; ex quo ejuſmodi expreſſiones $P^{\frac{m}{n}}$, exiſtente n numero pari, eodem jure Functionibus biformibus accenſeri poterunt: ſiquidem fractio $\frac{m}{n}$ ad minores terminos non fuerit reducibilis.

16. *Si fuerit* y *Functio quæcunque ipſius* z *; tum viciſſim* z *erit Functio ipſius* y.

Cum enim y ſit Functio ipſius z, ſive uniformis ſive multiformis; dabitur æquatio, qua y per z & conſtantes quantitates definitur. Ex eadem vero æquatione viciſſim z per y & conſtantes definiri poterit; unde quoniam y eſt quantitas variabilis, z æquabitur expreſſioni ex y & conſtantibus compoſitæ, eritque adeo Functio ipſius y. Hinc quoque patebit quam multifomis Functio futura ſit z ipſius y: fierique poteſt ut, etiamſi y fuerit Functio uniformis ipſius z, tamen z futura ſit Functio multiformis ipſius y. Sic ſi y ex hac æquatione per z definiatur; $y^3 = ayz - bzz$; erit utique y Functio triformis ipſius z, contra vero z Functio tantum biformis ipſius y.

17. *Si fuerint* y *&* x *Functiones ipſius* z *, erit quoque* y *Functio ipſius* x *, & viciſſim* x *Functio ipſius* y.

Cum enim ſit y Functio ipſius z, erit quoque z Functio ipſius y: ſimilique modo erit etiam z Functio ipſius x. Hanc ob rem Functio ipſius y æqualis erit Functioni ipſius x; ex qua æquatione & y per x & vice verſa x per y definiri poterit: quocirca manifeſtum eſt eſſe y Functionem ipſius x, atque x Functionem

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ipſius y. Sæpiſſime quidem has Functiones explicite exhibere non licet ob defectum Algebræ; interim tamen nihilo minus, quaſi omnes æquationes reſolvi poſſent, hæc Functionum reciprocatio perſpicitur. Ceterum per methodum in Algebra traditam, ex datis binis æquationibus, quarum altera continet y & z, altera vero x & z, per eliminationem quantitatis z formabitur una æquatio relationem inter x & y exprimens. CAP. I.

18. *Species denique quædam Functionum peculiares ſunt notandæ; ſic Functio par ipſius* z *eſt, quæ eundem dat valorem, ſive pro* z *ponatur valor determinatus* $+$ k *ſive* $-$ k.

Hujuſmodi ergo Functio par ipſius z erit zz; ſive enim ponatur $z = +k$, ſive $z = -k$, eundem valorem præbebit expreſſio zz, nempe $zz = +kk$. Simili modo Functiones pares ipſius z erunt hæ ipſius z poteſtates z^4, z^6, z^8, & generatim omnis poteſtas z^m, ſi fuerit m numerus par, ſive affirmativus ſive negativus. Quin etiam cum $z^{\frac{m}{n}}$ mentiatur Functionem ipſius z uniformem, ſi n ſit numerus impar, perſpicuum eſt $z^{\frac{m}{n}}$ fore Functionem parem ipſius z, ſi m fuerit numerus par, n vero numerus impar. Hanc ob rem, expreſſiones ex hujuſmodi poteſtatibus utcunque compoſitæ præbebunt Functiones pares ipſius z; ſic Z erit Functio par ipſius z, ſi fuerit $Z = a + bz^2 + cz^4 + dz^6 + \&c.$ item ſi fuerit $Z = \frac{a + bz^2 + cz^4 + dz^6 + \&c.}{\alpha + \beta z^2 + \gamma z^4 + \delta z^6 + \&c.}$; Similique modo exponentes fractos ipſius z introducendo, erit Z Functio par ipſius z ſi fuerit $Z = a + bz^{\frac{2}{3}} + cz^{\frac{2}{5}} + dz^{\frac{4}{7}}$ &c. vel $Z = a + bz^{-\frac{2}{3}} + cz^{-\frac{4}{3}} + dz^{-\frac{2}{1}} + \&c.$ vel $Z = \frac{a + bz^{\frac{2}{1}} + cz^{-\frac{4}{3}} + dz^{\frac{8}{5}}}{\alpha + \beta z^{\frac{2}{5}} + \gamma z^{-\frac{2}{3}} + \delta z^{\frac{4}{7}}}$. Cujuſmodi expreſſiones, cum omnes ſint Functiones uniformes ipſius z, appellari poterunt Functiones pares uniformes ipſius z.

B 2

19. *Functio multiformis par ipſius* z *eſt, quæ etiam ſi pro quovis valore ipſius* z *plures exhibeat valores determinatos, tamen eoſdem valores præbet, ſive ponatur* $z = +k$, *ſive* $z = -k$.

Sit Z ejuſmodi Functio multiformis par ipſius z; quoniam natura Functionis multifomis exprimitur per æquationem inter Z & z, in qua Z tot habeat dimenſiones, quot varios valores complectatur; manifeſtum eſt Z fore Functionem multiformem parem, ſi in æquatione naturam ipſius Z exprimente quantitas variabilis z ubique pares habeat dimenſiones. Sic, ſi fuerit $Z^2 = az^2 Z + bz^4$, erit Z Functio biformis par ipſius z; ſin autem ſit $Z^3 - az^2 Z^2 + bz^4 Z - cz^6 = 0$, erit Z Functio triformis par ipſius z; atque generatim, ſi P, Q, R, S, &c. denotent Functiones uniformes pares ipſius z, erit Z Functio biformis par ipſius z ſi ſit $Z^2 - PZ + Q = 0$. At Z erit Functio triformis par ipſius z ſi ſit $Z^3 - PZ^2 + QZ - R = 0$, & ita porro.

20. *Functio ergo, ſive uniformis ſive multiformis, par ipſius* z *erit ejuſmodi expreſſio ex quantitate variabili* z *& conſtantibus conflata, in qua ubique numerus dimenſionum ipſius* z *ſit par.*

Hujuſmodi ergo Functiones, præter uniformes quarum exempla ante ſunt allata, erunt hæ expreſſiones $a + \sqrt{(bb - zz)}$; $azz + \sqrt[3]{(a^4 z^4 - bz^2)}$ item $az^{\frac{2}{3}} + \sqrt[3]{(z^2 + \sqrt{(a^4 - z^4)})}$ &c.

Unde patet Functiones pares ita definiri poſſe, ut dicantur eſſe Functiones ipſius zz.

Sic enim ponatur $y = zz$, fueritque Z Functio quæcunque ipſius z; reſtituto ubique zz loco y, erit Z ejuſmodi Functio ipſius z, in qua z ubique parem habeat dimenſionum numerum. Excipiendi tamen ſunt ii caſus, quibus in expreſſione ipſius, Z occurunt $\sqrt{y}$: ac hujuſmodi aliæ formæ, quæ, facto $y = zz$, ſigna radicalia amittunt. Quamvis enim ſit $y + \sqrt{ay}$ Functio ipſius y, tamen poſito $y = zz$, eadem expreſſio non erit Functio par ipſius z; cum fiat $y + \sqrt{ay} = zz + z\sqrt{a}$. Excluſis ergo his caſibus, definitio ultima Functio-

num parium erit bona, atque ad ejuſmodi Functiones formandas idonea.

21. *Functio impar ipſius z eſt ejuſmodi Functio, cujus valor, ſi loco z ponatur — z, ſit quoque negativus.*

Hujuſmodi Functiones ergo impares erunt omnes poteſtates ipſius z, quarum exponentes ſunt numeri impares, ut z^1, z^3, z^5, z^7; &c. item z^{-1}, z^{-3}, z^{-5}; &c. tum vero etiam $z^{\frac{m}{n}}$ erit Functio impar, ſi ambo numeri, m & n fuerint numeri impares. Generatim vero omnis expreſſio ex hujuſmodi poteſtatibus compoſita erit Functio impar ipſius z; cujuſmodi ſunt, $az + bz^3$: $az + az^{-1}$; item $z^{\frac{1}{3}} + az^{\frac{3}{5}} + bz^{-\frac{1}{3}}$; &c. Harum autem Functionum natura & inventio ex Functionibus paribus facilius perſpicietur.

22. *Si Functio par ipſius z multiplicetur per z vel per ejuſdem Functionem imparem quamcunque, productum erit Functio impar ipſius z.*

Sit P Functio par ipſius z, quæ idcirco manet eadem ſi loco z ponatur $-z$; quod ſi ergo in producto Pz, ponatur $-z$ loco z, prodibit $-Pz$; unde Pz erit Functio impar ipſius z. Sit jam P Functio par ipſius z, & Q Functio impar ipſius z; atque ex Definitione patet ſi loco z ponatur $-z$, valorem ipſius P manere eundem, at valorem ipſius Q abire in ſui negativum $-Q$; quare productum PQ, poſito $-z$ loco z, abibit in $-PQ$, hoc eſt in ſui negativum; eritque ideo PQ Functio impar ipſius z. Sic cum ſit $a + \sqrt{(aa + zz)}$ functio par, & z^3 Functio impar ipſius z, erit productum $az^3 + z^3\sqrt{(aa + zz)}$ Functio impar ipſius z; ſimilique modo $z \times \frac{a + bzz}{\alpha + \beta zz} = \frac{az + bz^3}{\alpha + \beta zz}$ Functio impar ipſius z. Ex his vero etiam intelligitur, ſi duarum Functionum P & Q, quarum altera P eſt par, altera Q, impar, altera per alteram dividatur, quotum fore Functionem imparem; erit ergo $\frac{P}{Q}$ itemque $\frac{Q}{P}$ Functio impar ipſius z.

19. *Functio multiformis par ipſius* z *eſt, quæ etiam ſi pro quovis valore ipſius* z *plures exhibeat valores determinatos, tamen eoſdem valores præbet, ſive ponatur* z $= + k$, *ſive* z $= - k$.

Sit Z ejuſmodi Functio multiformis par ipſius z; quoniam natura Functionis multifomis exprimitur per æquationem inter Z & z, in qua Z tot habeat dimenſiones, quot varios valores complectatur; manifeſtum eſt Z fore Functionem multiformem parem, ſi in æquatione naturam ipſius Z exprimente quantitas variabilis z ubique pares habeat dimenſiones. Sic, ſi fuerit $Z^2 = a z^4 Z + b z^2$, erit Z Functio biformis par ipſius z; ſin autem ſit $Z^3 - a z^2 Z^2 + b z^4 Z - c z^8 = 0$, erit Z Functio triformis par ipſius z; atque generatim, ſi P, Q, R, S, &c. denotent Functiones uniformes pares ipſius z, erit Z Functio biformis par ipſius z ſi ſit $Z^2 - P Z + Q = 0$. At Z erit Functio triformis par ipſius z ſi ſit $Z^3 - P Z^2 + Q Z - R = 0$, & ita porro.

20. *Functio ergo, ſive uniformis ſive multiformis, par ipſius* z *erit ejuſmodi expreſſio ex quantitate variabili* z *& conſtantibus conflata, in qua ubique numerus dimenſionum ipſius* z *ſit par.*

Hujuſmodi ergo Functiones, præter uniformes quarum exempla ante ſunt allata, erunt hæ expreſſiones $a + \sqrt{(bb - zz)}$; $a zz + \sqrt[3]{(a^6 z^4 - b z^2)}$ item $a z^{\frac{2}{3}} + \sqrt[3]{(z^2 + \sqrt{(a^4 - z^4)})}$ &c.

Unde patet Functiones pares ita definiri poſſe, ut dicantur eſſe Functiones ipſius zz.

Sic enim ponatur $y = zz$, fueritque Z Functio quæcunque ipſius z; reſtituto ubique zz loco y, erit Z ejuſmodi Functio ipſius z, in qua z ubique parem habeat dimenſionum numerum. Excipiendi tamen ſunt ii caſus, quibus in expreſſione ipſius, Z occurunt $\sqrt{y}$: ac hujuſmodi aliæ formæ, quæ, facto $y = zz$, ſigna radicalia amittunt. Quamvis enim ſit $y + \sqrt{a y}$ Functio ipſius y, tamen poſito $y = zz$, eadem expreſſio non erit Functio par ipſius z; cum fiat $y + \sqrt{a y} = zz + z \sqrt{a}$. Excluſis ergo his caſibus, definitio ultima Functio-

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num parium erit bona, atque ad ejuſmodi Functiones formandas idonea.

21. *Functio impar ipſius z eſt ejuſmodi Functio, cujus valor, ſi loco z ponatur — z, ſit quoque negativus.*

Hujuſmodi Functiones ergo impares erunt omnes poteſtates ipſius z, quarum exponentes ſunt numeri impares, ut z^1, z^3, z^5, z^7; &c. item z^{-1}, z^{-3}, z^{-5}; &c. tum vero etiam $z^{\frac{m}{n}}$ erit Functio impar, ſi ambo numeri, m & n fuerint numeri impares. Generatim vero omnis expreſſio ex hujuſmodi poteſtatibus compoſita erit Functio impar ipſius z; cujuſmodi ſunt, $a z + b z^3 : a z + a z^{-1}$; item $z^{\frac{1}{3}} + a z^{\frac{3}{5}} + b z^{-\frac{1}{3}}$; &c. Harum autem Functionum natura & inventio ex Functionibus paribus facilius perſpicietur.

22. *Si Functio par ipſius z multiplicetur per z vel per ejuſdem Functionem imparem quamcunque, productum erit Functio impar ipſius z.*

Sit P Functio par ipſius z, quæ idcirco manet eadem ſi loco z ponatur $-z$; quod ſi ergo in producto Pz, ponatur $-z$ loco z, prodibit $-Pz$; unde Pz erit Functio impar ipſius z. Sit jam P Functio par ipſius z, & Q Functio impar ipſius z; atque ex Definitione patet ſi loco z ponatur $-z$, valorem ipſius P manere eundem, at valorem ipſius Q abire in ſui negativum $-Q$; quare productum PQ, poſito $-z$ loco z, abibit in $-PQ$, hoc eſt in ſui negativum; eritque ideo PQ Functio impar ipſius z. Sic cum ſit $a + \surd(aa + zz)$ functio par, & z^3 Functio impar ipſius z, erit productum $a z^3 + z^3 \surd(aa + zz)$ Functio impar ipſius z; ſimilique modo $z \times \frac{a + bzz}{\alpha + \beta zz} = \frac{az + bz^3}{\alpha + \beta zz}$ Functio impar ipſius z. Ex his vero etiam intelligitur, ſi duarum Functionum P & Q, quarum altera P eſt par, altera Q, impar, altera per alteram dividatur, quotum fore Functionem imparem; erit ergo $\frac{P}{Q}$ itemque $\frac{Q}{P}$ Functio impar ipſius z.

LIB. I. 23. *Si Functio impar per Functionem imparem vel multiplicetur, vel dividatur; quod reſultat erit Functio par.*

Sint Q & S Functiones impares ipſius z; ita ut, poſito $-z$ loco z, Q abeat in $-Q$, & S in $-S$; atque perſpicuum eſt tam productum QS, quam quotum $\frac{Q}{S}$ eundem valorem retinere, etiamſi pro z ponatur $-z$; ideoque eſſe utrumque Functionem parem ipſius z. Manifeſtum itaque porro eſt cujuſque Functionis imparis quadratum eſſe Functionem parem; cubum vero Functionem imparem; biquadratum iterum Functionem parem, atque ita porro.

24. *Si fuerit* y *Functio impar ipſius* z*; erit viciſſim* z *Functio impar ipſius* y.

Cum enim ſit y Functio impar ipſius z; ſi ponatur $-z$ loco z, abibit y in $-y$. Quod ſi ergo z per y definiatur, neceſſe eſt ut poſito $-y$ loco y, quoque z abeat in $-z$; eritque ideo z Functio impar ipſius y. Sic quia, poſito $y = z^3$, eſt y Functio impar ipſius z; erit quoque, ex æquatione $z^3 = y$ ſeu $z = y^{\frac{1}{3}}$, z Functio impar ipſius y. Et quia ſi fuerit $y = az + bz^3$, eſt y Functio impar ipſius z, erit viciſſim, ex æquatione $bz^3 + az = y$, valor ipſius z per y expreſſus Functio impar ipſius y.

25. *Si natura Functionis* y *per ejuſmodi æquationem definiatur, in cujus ſingulis terminis numerus dimenſionum, quas* y & z *occupant conjunctim, ſit vel par ubique, vel impar; tum erit* y *Functio impar ipſius* z.

Quod ſi enim in ejuſmodi æquatione ubique loco z ſcribatur $-z$; ſimulque $-y$ loco y; omnes æquationis termini vel manebunt iidem, vel fient negativi, utroque vero caſu æquatio manebit eadem. Unde patet $-y$ eodem modo per $-z$ determinatum iri, quo $+y$ per $+z$ determinatur; & hanc ob rem, ſi loco z ponatur $-z$, valor ipſius y abibit in $-y$, ſeu y erit Functio impar ipſius z. Sic ſi fuerit vel $yy = ayz + bzz + c$; vel $y^3 + ayyz = byzz + cy + dz$, ex utraque æquatione y erit Functio impar ipſius z.

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26. *Si* Z *fuerit Functio ipſius* z , *&* Y *Functio ipſius* y , *atque* Y *eodem modo definiatur per variabilem* y *& conſtantes , quo* Z *definitur per variabilem* z *& conſtantes ; tum hæ Functiones* Y *et* Z *vocantur Functiones ſimiles ipſarum* y *&* z.

Si ſcilicet fuerit $Z = a + bz + cz^2$, & $Y = a + by + cy^2$, erunt Z & Y Functiones ſimiles ipſarum z & y, ſimilique modo in multiformibus, ſi fuerit $Z^3 = azz\,Z + b$ & $Y^3 = ayy\,Y + b$; erunt Z & Y Functiones ſimiles ipſarum z & y. Hinc ſequitur, ſi Y & Z fuerint hujuſmodi Functiones ſimiles ipſarum y & z, tum ſi loco z ſcribatur y, Functionem Z abituram eſſe in Functionem Y. Solet hæc ſimilitudo etiam hoc modo verbis exprimi, ut Y talis Functio dicatur ipſius y, qualis Functio ſit Z ipſius z. Hæ locutiones perinde occurrent, ſive quantitates variabiles z & y a ſe invicem pendeant, ſive ſecus: ſic qualis Functio eſt $ay + by^3$ ipſius y, talis Functio erit $a(y + n) + b(y + n)^3$ ipſius $y + n$, exiſtente ſcilicet $z = y + n$: tum qualis Functio eſt $\frac{a + bz + czz}{\alpha + \beta z + \gamma zz}$ ipſius z, talis Functio erit $\frac{azz + bz + c}{\alpha zz + \beta z + \gamma}$ ipſius $\frac{1}{z}$; poſito $y = \frac{1}{z}$. Atque ex his luculenter perſpicitur ratio ſimilitudinis Functionum, cujus per univerſam Analyſin ſublimiorem uberrimus eſt uſus. Ceterum hæc in genere de natura Functionum unius variabilis ſufficere poſſunt; cum plenior expoſitio in applicatione ſequente tradatur.

CAPUT II.

De transformatione Functionum.

27. F*UNCTIONES in alias formas tranſmutantur , vel loco quantitatis variabilis aliam introducendo , vel eandem quantitatem variabilem retinendo.*

Quod ſi eadem quantitas variabilis ſervatur, Functio proprie mutari non poteſt. Sed omnis transformatio conſiſtit in

LIB. I. alio modo eandem Functionem exprimendi, quemadmodum ex Algebra conſtat eandem quantitatem per plures diverſas formas exprimi poſſe. Hujuſmodi transformationes ſunt, ſi loco hujus Functionis $2 - 3z + zz$ ponatur $(1 - z)(2 - z)$, vel $(a + z)^3$ loco $a^3 + 3aaz + 3azz + z^3$, vel $\frac{a}{a-z} + \frac{a}{a+z}$ loco $\frac{2aa}{aa - zz}$; vel $\sqrt{(1 + zz)} + z$ loco $\frac{1}{\sqrt{(1+zz)} - z}$; quæ expreſſiones, etſi forma differunt, tamen revera congruunt. Sæpe numero autem harum plurium formarum idem ſignificantium una aptior eſt ad propoſitum efficiendum quam reliquæ, & hanc ob rem formam commodiſſimam eligi oportet.

Alter transformationis modus, quo loco quantitatis variabilis z alia quantitas variabilis y introducitur, quæ quidem ad z datam teneat relationem, per ſubſtitutionem fieri dicitur; hocque modo ita uti convenit, ut Functio propoſita ſuccinctius & commodius exprimatur, uti ſi iſta propoſita fuerit ipſius z Functio, $a^4 - 4a^3z + 6aazz - 4az^3 + z^4$; ſi loco $a - z$ ponatur y, prodibit iſta multo ſimplicior ipſius y Functio y^4: &, ſi habeatur hæc Functio irrationalis $\sqrt{(aa + zz)}$ ipſius z, ſi ponatur $z = \frac{aa - yy}{2y}$, iſta Functio per y expreſſa fiet rationalis $= \frac{aa + yy}{2y}$. Hunc autem transformationis modum in ſequens Caput differam, hoc Capite illum, qui ſine ſubſtitutione procedit, expoſiturus.

29. *Functio integra ipſius z ſæpenumero commode in ſuos factores reſolvitur, ſicque in productum transformatur.*

Quando Functio integra hoc pacto in factores reſolvitur, ejus natura multo facilius perſpicitur; caſus enim ſtatim innoteſcunt, quibus Functionis valor fit $= 0$. Sic hæc ipſius z Functio $6 - 7z + z^3$ transformatur in hoc productum $(1 - z)(2 - z)(3 + z)$ ex quo ſtatim liquet Functionem propoſitam tribus caſibus fieri $= 0$; ſcilicet ſi $z = 1$, & $z = 2$, & $z = -3$, quæ proprietates ex forma $6 - 7z + z^3$ non tam facile intelliguntur. Iſtiuſmodi factores, in quibus variabilis z nulla

nulla occurrit poteſtas quadrata vel altior, vocantur Factores ſimplices, ut diſtinguantur a Factoribus compoſitis, in quibus ipſius z ineſt quadratum vel cubus, vel alia poteſtas altior. Erit ergo in genere $f+gz$ forma Factorum ſimplicium, $f+gz+hzz$ forma Factorum duplicium, $f+gz+hzz+iz^3$ forma Factorum triplicium, & ita porro. Perſpicuum autem eſt Factorem duplicem duos complecti valores ſimplices, Factorem triplicem tres ſimplices, & ita porro. Hinc Functio ipſius z integra, in qua exponens ſummæ poteſtatis ipſius z eſt $=n$, continebit n Factores ſimplices; ex quo ſimul, ſi qui Factores fuerint vel duplices vel triplices, &c. numerus Factorum cognoſcetur. CAP. II.

29. *Factores ſimplices Functionis cujuſcunque integræ* Z *ipſius* z *reperiuntur, ſi Functio* Z *nihilo æqualis ponatur, atque ex hac æquatione omnes ipſius* z *radices inveſtigentur: ſingulæ enim ipſius* z *radices dabunt totidem Factores ſimplices Functionis* Z.

Quod ſi enim ex æquatione $Z=0$, fuerit quæpiam radix $z=f$, erit $z-f$ diviſor, ac proinde Factor Functionis Z, ſic igitur inveſtigandis omnibus radicibus æquationis $Z=0$, quæ ſint $z=f$, $z=g$, $z=h$; &c., Functio Z reſolvetur in ſuos Factores ſimplices, atque transformabitur in productum $Z=(z-f)(z-g)(z-h)$ &c.: ubi quidem notandum eſt ſi ſummæ poteſtatis ipſius z in Z non fuerit coefficiens $=+1$, tum productum $(z-f)(z-g)$ &c. inſuper per illum coefficientem multiplicari debere. Sic ſi fuerit $Z=Az^n+Bz^{n-1}+Cz^{n-2}+$ &c. erit $Z=A(z-f)(z-g)(z-h)$ &c. At ſi fuerit $Z=A+Bz+Cz^2+Dz^3+Ez^4+$ &c. atque æquationis $Z=0$ radices z repertæ ſint; f; g; h; i; &c. erit $Z=A(1-\frac{z}{f})(1-\frac{z}{g})(1-\frac{z}{h})$ &c. Ex his autem viciſſim intelligitur, ſi Functionis Z Factor fuerit $z-f$, ſeu $1-\frac{z}{f}$; tum valorem Functionis in nihilum abire, ſi loco z ponatur f. Facto enim $z=f$, unus Factor $z-f$, ſeu $1-\frac{z}{f}$ Functionis Z, ideoque ipſa Functio Z evaneſcere debet.

30. *Factores ſimplices ergo erunt vel reales, vel imaginarii; & , ſi Functio Z habeat Factores imaginarios, eorum numerus ſemper erit par.*

Cum enim Factores ſimplices naſcantur ex radicibus æquationis $Z = 0$, radices reales præbebunt Factores reales, & imaginariæ imaginarios; in omni autem æquatione numerus radicum imaginariarum ſemper eſt par: quamobrem Functio Z, vel nullos habebit Factores imaginarios, vel duos, vel quatuor, vel ſex, &c. Quod ſi Functio Z duos tantum habeat Factores imaginarios, eorum productum erit reale, ideoque præbebit Factorem duplicem realem. Sit enim $P =$ producto ex omnibus Factoribus realibus, erit productum duorum Factorum imaginariorum $= \frac{Z}{P}$; hincque reale. Simili modo ſi Functio Z habeat quatuor, vel ſex, vel octo, &c. Factores imaginarios; erit eorum productum ſemper reale: nempe æquale quoto, qui oritur, ſi Functio Z dividatur per productum omnium Factorum realium.

31. *Si fuerit Q productum reale ex quatuor Factoribus ſimplicibus imaginariis, tum idem hoc productum Q reſolvi poterit in duos Factores duplices reales.*

Habebit enim Q ejuſmodi formam $z^4 + Az^3 + Bz^2 + Cz + D$; quæ ſi negetur in duos Factores duplices reales reſolvi poſſe, reſolubilis erit ſtatuenda in duos Factores duplices imaginarios, qui hujuſmodi formam habebunt $zz - 2(p + q\sqrt{-1})z + r + s\sqrt{-1}$, & $zz - 2(p - q\sqrt{-1})z + r - s\sqrt{-1}$; aliæ enim formæ imaginariæ concipi non poſſunt, quarum productum fiat reale, nempe $= z^4 + Az^3 + Bz^2 + Cz + D$. Ex his autem Factoribus imaginariis duplicibus ſequentes emergent quatuor Factores ſimplices imaginarii ipſius Q,

I. $z - (p + q\sqrt{-1}) + \sqrt{(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1})}$
II. $z - (p + q\sqrt{-1}) - \sqrt{(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1})}$
III. $z - (p - q\sqrt{-1}) + \sqrt{(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1})}$
IV. $z - (p - q\sqrt{-1}) - \sqrt{(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1})}$

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Horum Factorum multiplicentur primus ac tertius in ſe invicem, poſito brevitatis gratia, $t = pp - qq - r$, & $u = 2pq - s$; eritque horum Factorum productum $= zz - (2p - \sqrt{2t + 2\sqrt{(tt+uu)}})z$

$$+ pp + qq - p\sqrt{2t + 2\sqrt{(tt+uu)}} + \sqrt{(tt+uu)};$$
$$+ q\sqrt{-2t + 2\sqrt{(tt+uu)}}$$

quod utique eſt reale. Simili autem modo productum ex Factoribus ſecundo & quarto erit reale nempe $= zz - (2p + \sqrt{2t + 2\sqrt{tt+uu}})z$

$$+ pp + qq + p\sqrt{2t + 2\sqrt{(tt+uu)}} + \sqrt{(tt+uu)}.$$
$$+ q\sqrt{-2t + 2\sqrt{(tt+uu)}}$$

Quocirca productum propoſitum Q, quod in duos Factores duplices reales reſolvi poſſe negabatur, nihilo minus actu in duos Factores duplices reales eſt reſolutum.

32. *Si Functio integra Z ipſius z quotcunque habeat Factores ſimplices imaginarios, bini ſemper ita conjungi poſſunt, ut eorum productum fiat reale.*

Quoniam numerus radicum imaginariarum ſemper eſt par, ſit is $= 2n$; ac primo quidem patet productum harum radicum imaginarium omnium eſſe reale. Quod ſi ergo duæ tantum radices imaginariæ habeantur, erit earum productum utique reale; ſin autem quatuor habeantur Factores imaginarii, tum, uti vidimus, eorum productum reſolvi poteſt in duos Factores duplices reales formæ $fzz + gz + h$. Quanquam autem eundem demonſtrandi modum ad altiores poteſtates extendere non licet, tamen extra dubium videtur eſſe poſitum eandem proprietatem in quotcumque Factores imaginarios competere; ita ut ſemper loco $2n$ Factorum ſimplicium imaginariorum induci queant n Factores duplices reales. Hinc omnis Functio integra ipſius z reſolvi poterit in Factores reales vel ſimplices vel duplices. Quod quamvis non ſummo rigore ſit demonſtratum, tamen ejus veritas in ſequentibus magis corroborabitur, ubi hujus generis Functiones $a + bz^n$; $a + bz^n + cz^{2n}$; $a + bz^n + cz^{2n} + dz^{3n}$ &c. actu in iſtiuſmodi Factores duplices reales reſolventur.

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LIB. I. 33. *Si Functio integra* Z, *poſito* $z = a$, *induat valorem* A; *& poſito* $z = b$, *induat valorem* B; *tum, loco* z *valores medios inter* a *&* b *ponendo, Functio* Z *quoſvis valores medios inter* A *&* B *accipere poteſt.*

Cum enim Z ſit Functio uniformis ipſius z, quicunque valor realis ipſi z tribuatur, Functio quoque Z hinc valorem realem obtinebit. Cum igitur Z, priore caſu $z = a$, nanciſcatur valorem A; poſteriore caſu $z = b$, autem, valorem B; ab A ad B tranſire non poterit, niſi per omnes valores medios tranſeundo. Quod ſi ergo æquatio $Z - A = 0$ habeat radicem realem, ſimulque $Z - B = 0$ radicem realem ſuppeditet; tum æquatio quoque $Z - C = 0$ radicem habebit realem; ſi quidem C intra valores A & B contineatur. Hinc ſi expreſſiones $Z - A$ & $Z - B$ habeant Factorem ſimplicem realem, tum expreſſio quæcunque $Z - C$ Factorem ſimplicem habebit realem, dummodo C intra valores A & B contineatur.

34. *Si in Functione integra* Z *exponens maximæ ipſius* z *poteſtatis fuerit numerus impar* $2n + 1$, *tum ea Functio* Z *unicum ad minimum habebit Factorem ſimplicem realem.*

Habebit ſcilicet Z hujuſmodi formam $z^{2n+1} + \alpha z^{2n} + \beta z^{2n-1} + \gamma z^{2n-2} +$ &c. in qua ſi ponatur $z = \infty$, quia valores ſingulorum terminorum præ primo evaneſcunt, fiet $Z = (\infty)^{2n+1} = \infty$; ideoque $Z - \infty$ Factorem ſimplicem habebit realem nempe $z - \infty$. Sin autem ponatur $z = -\infty$, fiet $Z = (-\infty)^{2n+1} = -\infty$, ideoque habebit $Z + \infty$ Factorem ſimplicem realem $z + \infty$. Cum igitur tam $Z - \infty$, quam $Z + \infty$ habeat Factorem ſimplicem realem; ſequitur etiam $Z + C$ habiturum eſſe Factorem ſimplicem realem, ſiquidem C contineatur intra limites $+\infty$ & $-\infty$; hoc eſt ſi C fuerit numerus realis quicunque, ſive affirmativus, ſive negativus. Hanc ob rem, facto $C = 0$, habebit quoque ipſa Functio Z Factorem ſimplicem realem $z - c$; atque quantitas c

continebitur intra limites $+\infty$ & $-\infty$, eritque idcirco vel quantitas affirmativa, vel negativa, vel nihil.

35. *Functio igitur integra Z, in qua exponens maximæ poteſtatis ipſius z eſt numerus impar, vel unum habebit Factorem ſimplicem realem, vel tres, vel quinque, vel ſeptem, &c.*

Cum enim demonſtratum ſit Functionem Z certo unum habere Factorem ſimplicem realem $z-c$; ponamus eam præterea unum Factorem habere $z-d$, atque dividatur Functio Z, in qua maxima ipſius z poteſtas ſit z^{2n+1}, per $(z-c)(z-d)$, erit quoti maxima poteſtas $= z^{2n-1}$, cujus exponens, cum ſit numerus impar, indicat denuo ipſius Z dari Factorem ſimplicem realem. Si ergo Z plures uno habeat Factores ſimplices reales, habebit vel tres, vel (quoniam eodem modo progredi licet) quinque, vel ſeptem, &c. Erit ſcilicet numerus Factorum ſimplicium realium impar, & quia numerus omnium Factorum ſimplicium eſt $=2n+1$, erit numerus Factorum imaginariorum par.

36. *Functio integra Z, in qua exponens maximæ poteſtatis ipſius z eſt numerus par 2n, vel duos habebit Factores ſimplices reales, vel quatuor, vel ſex, vel &c.*

Ponamus ipſius Z conſtare Factorum ſimplicium realium numerum imparem $2m+1$; ſi ergo per horum omnium productum dividatur Functio Z, quoti maxima poteſtas erit $= z^{2n-2m-1}$, ejuſque ideo exponens numerus impar; habebit ergo Functio Z præterea unum certo Factorem ſimplicem realem, ex quo numerus omnium Factorum ſimplicium realium ad minimum erit $=2m+2$, ideoque par; ac numerus Factorum imaginariorum pariter par. Omnis ergo Functionis integræ Factores ſimplices imaginarii ſunt numero pares; quemadmodum quidem jam ante ſtatuimus.

37. *Si in Functione integra Z exponens maximæ poteſtatis ipſius z fuerit numerus par, atque terminus abſolutus, ſeu conſtans, ſigno — affectus, tum Functio Z ad minimum duos habet Factores ſimplices reales.*

LIB. I. Functio ergo Z, de qua hic ſermo eſt, hujuſmodi formam habebit $z^{2n} \pm \alpha z^{2n-1} \pm \beta z^{2n-2} \pm \ldots\ldots \pm \nu z - A$. Si jam ponatur $z = \infty$, fiet, uti ſupra vidimus, $Z = \infty$; atque, ſi ponatur $z = 0$, fiet $Z = -A$. Habebit ergo $Z - \infty$ Factorem realem $z - \infty$, & $Z + A$ Factorem $z - 0$: unde cum o contineatur intra limites $-\infty$ & $+A$, ſequitur $Z + 0$ habere Factorem ſimplicem realem $z - c$, ita ut c contineatur intra limites o & ∞. Deinde, cum poſito $z = -\infty$, fiat $Z = \infty$, ideoque $Z - \infty$ Factorem habeat $z + \infty$, & $Z + A$ Factorem $z + 0$, ſequitur quoque $Z + 0$ Factorem ſimplicem realem habere $z + d$; ita ut d intra limites o & ∞ contineatur; unde conſtat propoſitum. Ex his igitur perſpicitur ſi Z talis fuerit Functio, qualis hic eſt deſcripta, æquationem $Z = 0$, duas ad minimum habere debere radices reales, alteram affirmativam, alteram negativam. Sic æquatio hæc $z^4 + \alpha z^3 + \beta z^2 + \gamma z - aa = 0$, duas habet radices reales, alteram affirmativam, alteram negativam.

38. *Si in Functione fracta, quantitas variabilis z tot vel plures habeat dimenſiones in numeratore, quam in denominatore; tum iſta Functio reſolvi poterit in duas partes, quarum altera eſt Functio integra, altera fracta, in cujus numeratore quantitas variabilis z pauciores habeat dimenſiones quam in denominatore.*

Si enim exponens maximæ poteſtatis ipſius z minor fuerit in denominatore quam in numeratore; tum numerator per denominatorem dividatur more ſolito, donec in quoto ad exponentes negativos ipſius z perveniatur; hoc ergo loco abrupta diviſionis operatione quotus conſtabit ex parte integra atque fractione, in cujus numeratore minor erit dimenſionum numerus ipſius z quam in denominatore; hic autem quotus Functioni propoſitæ eſt æqualis. Sic, ſi hæc propoſita fuerit Functio fracta $\frac{1+z^4}{1+zz}$, ea per diviſionem ita reſolvetur:

$$\begin{array}{r|l} zz+1) & z^4+1\ \left(zz-1+\frac{2}{1+zz}\right. \\ & \underline{z^4+zz} \\ & -zz+1 \\ & \underline{-zz-1} \\ & \qquad +2 \end{array}$$

eritque $\frac{1+z^4}{1+zz} = zz - 1 + \frac{2}{1+zz}$. Hujuſmodi Functiones fractæ, in quibus quantitas variabilis z tot vel plures habet dimenſiones in numeratore quam in denominatore, ad ſimilitudinem Arithmeticæ vocari poſſunt fractiones ſpuriæ, vel Functiones fractæ ſpuriæ, quo diſtinguantur a Functionibus fractis genuinis, in quarum numeratore quantitas variabilis z pauciores habet dimenſiones quam in denominatore. Functio itaque fracta ſpuria reſolvi poterit in Functionem integram, & Functionem fractam genuinam; hæcque reſolutio per vulgarem diviſionis operationem abſolvetur.

39. *Si denominator Functionis fractæ duos habeat Factores inter ſe primos; tum ipſa Functio fracta reſolvetur in duas fractiones, quarum denominatores ſint illis binis Factoribus reſpective æquales.*

Quanquam hæc reſolutio ad Functiones fractas ſpurias æque pertinet atque ad genuinas, tamen eam ad genuinas potiſſimum accommodabimus. Reſoluto autem denominatore hujuſmodi Functionis fractæ in duos Factores inter ſe primos, ipſa Functio reſolvetur in duas alias Functiones fractas genuinas, quarum denominatores ſint illis binis Factoribus reſpective æquales; hæcque reſolutio, ſi quidem fractiones ſint genuinæ, unico modo fieri poteſt; cujus rei veritas ex exemplo clarius quam per ratiocinium perſpicietur. Sit ergo propoſita hæc Functio fracta $\frac{1-2z+3zz-4z^3}{1+4z^4}$, cujus denominator $1+4z^4$ cum ſit æqualis huic producto $(1+2z+2zz)(1-2z+2zz)$, fractio propoſita in duas fractiones reſolvetur, quarum alterius denominator erit $1+2z+2zz$, alterius $1-2z+2zz$: ad quas inveniendas, quia ſunt genuinæ, ſtatuantur numeratores illius $= \alpha + \beta z$, hujus $= \gamma + \delta z$, eritque per hypotheſin

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 $\frac{1-2z+3zz-4z^3}{1+4z^4} = \frac{\alpha+\beta z}{1+2z+2zz} + \frac{\gamma+\delta z}{1-2z+2zz}$: addantur actu hæ duæ fractiones, eritque ſummæ

Numerator	Denominator
$+\alpha - 2\alpha z + 2\alpha zz$ $+\beta z - 2\beta zz + 2\beta z^3$ $+\gamma + 2\gamma z + 2\gamma zz$ $+\delta z + 2\delta zz + 2\delta z^3$	$1+4z^4$

Cum ergo denominator æqualis ſit denominatori fractionis propoſitæ, numeratores quoque æquales reddi debent: quod, ob tot litteras incognitas α, β, γ, δ, quot ſunt termini æquales efficiendi, utique fieri, idque unico modo poterit: nanciſcimur ſcilicet has quatuor æquationes

I. $\alpha + \gamma = 1$ III. $2\alpha - 2\beta + 2\gamma + 2\delta = 3$
II. $-2\alpha + \beta + 2\gamma + \delta = -2$ IV. $2\beta + 2\delta = -4$.

Hinc ob $\alpha + \gamma = 1$, & $\beta + \delta = -2$; æquationes II. & III. dabunt $\alpha - \gamma = 0$ & $\delta - \beta = \frac{1}{2}$; ex quibus fit $\alpha = \frac{1}{2}$; $\gamma = \frac{1}{2}$; $\beta = \frac{-5}{4}$; $\delta = \frac{-3}{4}$; ideoque fractio propoſita $\frac{1-2z+3zz-4z^3}{1+4z^4}$, transformatur in has duas $\frac{\frac{1}{2}-\frac{5}{4}z}{1+2z+2zz} + \frac{\frac{1}{2}-\frac{3}{4}z}{1-2z+2zz}$. Simili autem modo facile perſpicietur reſolutionem ſemper ſuccedere debere: quoniam ſemper tot litteræ incognitæ introducuntur, quot opus eſt ad numeratorem propoſitum eliciendum. Ex doctrina vero fractionum communi intelligitur hanc reſolutionem ſuccedere non poſſe, niſi iſti denominatoris Factores fuerint inter ſe primi.

40. *Functio igitur fracta* $\frac{M}{N}$ *in tot fractiones ſimplices formæ* $\frac{A}{p-qz}$ *reſolvi poterit, quot Factores ſimplices habet denominator* N *inter ſe inæquales.*

Repræſentat

Repræſentat hic fractio $\frac{M}{N}$ Functionem quamcunque fractam genuinam, ita ut M & N ſint Functiones integræ ipſius z, atque ſumma poteſtas ipſius z in M minor ſit quam in N. Quod ſi ergo denominator N in ſuos Factores ſimplices reſolvatur, hique inter ſe fuerint inæquales, expreſſio $\frac{M}{N}$ in tot fractiones reſolvetur, quot Factores ſimplices in denominatore N continentur; propterea quod quiſque Factor abit in denominatorem fractionis partialis. Si ergo $p - qz$ fuerit Factor ipſius N, is erit denominator fractionis cujuſdam partialis, &, cum in numeratore hujus fractionis numerus dimenſionum ipſius z minor eſſe debeat quam in denominatore $p - qz$, numerator neceſſario erit quantitas conſtans. Hinc ex unoquoque Factore ſimplici $p - qz$ denominatoris N naſcetur fractio ſimplex $\frac{A}{p - qz}$; ita ut ſumma omnium harum fractionum ſit æqualis fractioni propoſitæ $\frac{M}{N}$.

EXEMPLUM.

Sit, exempli cauſa, propoſita hæc Functio fracta $\frac{1 + zz}{z - z^3}$; quia Factores ſimplices denominatoris ſunt z, $1 - z$, & $1 + z$, iſta Functio reſolvetur in has tres fractiones ſimplices $\frac{A}{z} + \frac{B}{1 - z} + \frac{C}{1 + z} = \frac{1 + zz}{z - z^3}$; ubi numeratores conſtantes A, B, & C definire oportet. Reducantur hæ fractiones ad communem denominatorem, qui erit $z - z^3$; atque numeratorum ſumma æquari debebit ipſi $1 + zz$, unde iſta æquatio oritur:

$$\begin{aligned} A + Bz - Azz &= 1 + zz = 1 + 0z + zz. \\ + Cz + Bzz & \\ - Czz & \end{aligned}$$

 quæ totidem comparationes præbet, quot ſunt litteræ incognitæ A, B, C; erit ſcilicet,

I°. $A = 1$.
II°. $B + C = 0$.
III°. $-A + B - C = 1$:

Hinc fit $B - C = 2$; & porro $A = 1$; $B = 1$ & $C = -1$. Functio ergo propoſita $\frac{1+zz}{z-z^3}$ reſolvitur in hanc formam $\frac{1}{z} + \frac{1}{1-z} - \frac{1}{1+z}$. Simili autem modo intelligitur, quotcunque habuerit denominator N Factores ſimplices inter ſe inæquales, ſemper fractionem $\frac{M}{N}$ in totidem fractiones ſimplices reſolvi. Sin autem aliquot Factores fuerint æquales inter ſe, tum alio modo poſt-explicando reſolutio inſtitui debet.

41. *Cum igitur quilibet Factor ſimplex denominatoris* N *ſuppeditet fractionem ſimplicem pro reſolutione Functionis propoſitæ* $\frac{\mathrm{M}}{\mathrm{N}}$; *oſtendendum eſt quomodo ex Factore ſimplice denominatoris* N *cognito, fractio ſimplex reſpondens reperiatur.*

Sit $p - qz$ Factor ſimplex ipſius N, ita ut ſit $N = (p - qz)S$; atque S Functio integra ipſius z; ponatur fractio ex Factore $p - qz$ orta $= \frac{A}{p - qz}$, & ſit fractio ex altero Factore denominatoris S oriunda $= \frac{P}{S}$, ita ut, ſecundum §. 39., futurum ſit $\frac{M}{N} = \frac{A}{p-qz} + \frac{P}{S} = \frac{M}{(p-qz)S}$; hinc erit $\frac{P}{S} = \frac{M - AS}{(p-qz)S}$; quæ fractiones cum congruere debeant, neceſſe eſt ut $M - AS$ ſit diviſibile per $p - qz$; quoniam Functio integra P ipſi quoto æquatur. Quando vero $p - qz$ Diviſor exiſtit ipſius $M - AS$, hæc expreſſio poſito $z = \frac{p}{q}$ evaneſcit. Ponatur ergo ubique loco z hic valor conſtans $\frac{p}{q}$ in M

& S, erit $M - AS = 0$, ex quo fiet $A = \frac{M}{S}$; hocque ergo modo reperitur numerator A fractionis quæſitæ $\frac{A}{p - qz}$; atque ſi ex ſingulis denominatoris N Factoribus ſimplicibus, dummodo ſint inter ſe inæquales, hujuſmodi fractiones ſimplices formentur, harum fractionum ſimplicium omnium ſumma erit æqualis Functioni propoſitæ $\frac{M}{N}$.

EXEMPLUM.

Sic, ſi in Exemplo præcedente $\frac{1 + zz}{z - z^3}$, ubi eſt $M = 1 + zz$, & $N = z - z^3$, ſumatur z pro Factore ſimplice, erit $S = 1 - zz$, atque fractionis ſimplicis $\frac{A}{z}$ hinc ortæ erit numerator $A = \frac{1 + zz}{1 - zz} = 1$ poſito $z = 0$, quem valorem z obtinet ſi ipſe hic Factor ſimplex z nihilo æqualis ponatur. Simili modo ſi pro denominatoris Factore ſumatur $1 - z$, ut ſit $S = z + zz$ erit $A = \frac{1 + zz}{z + zz}$, facto $1 - z = 0$, unde erit $A = 1$, & ex Factore $1 - z$ naſcitur fractio $\frac{1}{1 - z}$. Tertius denique Factor $1 + z$, ob $S = z - zz$, & $A = \frac{1 + zz}{z - zz}$, poſito $1 + z = 0$, ſeu $z = -1$, dabit $A = -1$, & fractionem ſimplicem $= \frac{-1}{1 + z}$. Quare per hanc regulam reperitur $\frac{1 + zz}{z - z^3} = \frac{1}{z} + \frac{1}{1 - z} - \frac{1}{1 + z}$, ut ante.

42. *Functio fracta hujus formæ* $\frac{P}{(p - qz)^n}$, *cujus numerator* P *non tantam ipſius* z *poteſtatem involvit quantam denominator* $(p - qz)^n$, *tranſmutari poteſt in hujuſmodi fractiones partiales*

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$$\frac{A}{(p-qz)^n}+\frac{B}{(p-qz)^{n-1}}+\frac{C}{(p-qz)^{n-2}}+\ldots\ldots+\frac{K}{p-qz},$$

quarum omnium numeratores ſint quantitates conſtantes.

Quoniam maxima poteſtas ipſius z in P minor eſt quam z^n, erit z^{n-1}, ideoque P hujuſmodi habebit formam:

$$\alpha+\beta z+\gamma z^2+\delta z^3+\ldots\ldots\ldots\ldots+\varkappa z^{n-1}$$

exiſtente terminorum numero $= n$, cui æquari debet numerator ſummæ omnium fractionum partialium, poſtquam ſingulæ ad eundem denominatorem $(p-qz)^n$ fuerint perductæ: qui numerator propterea erit $= A+B(p-qz)+C(p-qz)^2 + D(p-qz)^3+\ldots\ldots+K(p-qz)^{n-1}$. Hujus maxima ipſius z poteſtas eſt, ut ibi, z^{n-1}, atque tot habentur litteræ incognitæ $A, B, C, \ldots\ldots K$, (quarum numerus eſt $= n$,) quot ſunt termini congruentes reddendi. Quamobrem litteræ conſtantes A, B, C, &c. ita definiri poterunt, ut fiat Functio fracta genuina $\frac{P}{(p-qz)^n}=\frac{A}{(p-qz)^n}+\frac{B}{(p-qz)^{n-1}}+\frac{C}{(p-qz)^{n-2}}+\frac{D}{(p-qz)^{n-3}}+\ldots\ldots\ldots+\frac{K}{p-qz}$. Ipſa autem horum numeratorum inventio mox facilis aperietur.

43. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat* $(p-qz)^2$, *ſequenti modo fractiones partiales ex hoc Factore oriundæ reperientur.*

Cujuſmodi fractiones partiales ex ſingulis Factoribus denominatoris ſimplicibus, qui alios ſibi æquales non habeant, oriantur, ante eſt oſtenſum: nunc igitur ponamus duos Factores inter ſe eſſe æquales, ſeu, iis conjunctis, denominatoris N Factorem eſſe $(p-qz)^2$. Ex hoc ergo Factore per §. præced. duæ naſcentur fractiones partiales hæ $\frac{A}{(p-qz)^2}+\frac{B}{p-qz}$. Sit au-

tem $N = (p - qz)^2 S$, eritque $\frac{M}{N} = \frac{M}{(p-qz)^2 S} = \frac{A}{(p-qz)^2}$ $+ \frac{B}{p-qz} + \frac{P}{S}$, denotante $\frac{P}{S}$ omnes fractiones ſimplices junctim ſumptas ex denominatoris Factore S ortas. Hinc erit $\frac{P}{S}$ $= \frac{M - AS - B(p-qz)S}{(p-qz)^2 S}$, & $P = \frac{M - AS - B(p-qz)S}{(p-qz)^2}$ $=$ Functioni integræ. Debet ergo $M - AS - B(p-qz)S$ diviſibile eſſe per $(p-qz)^2$: ſit primum diviſibile per $p - qz$, atque tota expreſſio $M - AS - B(p-qz)S$ evaneſcet, poſito $p - qz = 0$, ſeu $z = \frac{p}{q}$; ponatur ergo ubique $\frac{p}{q}$ loco z, eritque $M - AS = 0$, ideoque $A = \frac{M}{S}$; ſcilicet fractio $\frac{M}{S}$, ſi loco z ubique ponatur $\frac{p}{q}$, dabit valorem ipſius A conſtantem. Hoc invento quantitas $M - AS - B(p-qz)S$ etiam per $(p-qz)^2$ diviſibilis eſſe debet, ſeu $\frac{M - AS}{p-qz} -$ BS denuo per $p - qz$ diviſibile eſſe debet. Poſito ergo ubique $z = \frac{p}{q}$ erit $\frac{M - AS}{p - qz} = BS$, ideoque $B = \frac{M - AS}{(p-qz)S}$ $= \frac{1}{p-qz}\left(\frac{M}{S} - A\right)$, ubi notandum eſt, cum $M - AS$ diviſibile ſit per $p - qz$, hanc diviſionem prius inſtitui debere, quam loco z ſubſtituatur $\frac{p}{q}$. Vel ponatur $\frac{M - AS}{p - qz} = T$, eritque $B = \frac{T}{S}$ poſito $z = \frac{p}{q}$; inventis ergo numeratoribus A & B, erunt fractiones partiales ex denominatoris N Factore $(p - qz)^2$ ortæ hæ $\frac{A}{(p-qz)^2} + \frac{B}{p-qz}$.

EXEMPLUM I.

Sit hæc propoſita Functio fracta $\frac{1 - zz}{zz(1 + zz)}$ erit, ob de-

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LIB. I. nominatoris Factorem quadratum zz; $S = 1 + zz$ & $M = 1 - zz$. Sint fractiones partiales ex zz ortæ $\frac{A}{zz} + \frac{B}{z}$, erit $A = \frac{M}{S} = \frac{1 - zz}{1 + zz}$, poſito Factore $z = 0$; hincque $A = 1$. Tum erit $M - AS = -2zz$ quod diviſum per Factorem ſimplicem z, dabit $T = -2z$, hincque $B = \frac{T}{S} = \frac{-2z}{1 + zz}$, poſito $z = 0$; unde erit $B = 0$; atque ex Factore denominatoris zz orietur unica hæc fractio partialis $\frac{1}{zz}$.

EXEMPLUM II.

Sit hæc propoſita Functio fracta $\frac{z^3}{(1-z)^2(1+z^4)}$, cujus, ob denominatoris Factorem quadratum $(1-z)^2$, fractiones partiales ſint $\frac{A}{(1-z)^2} + \frac{B}{1-z}$. Erit ergo $M = z^3$ & $S = 1 + z^4$; ideoque $A = \frac{M}{S} = \frac{z^3}{1+z^4}$, poſito $1 - z = 0$, ſeu $z = 1$: unde fit $A = \frac{1}{2}$. Prodibit ergo $M - AS = z^3 - \frac{1}{2} - \frac{1}{2}z^4 = -\frac{1}{2} + z^3 - \frac{1}{2}z^4$, quod per $1 - z$ diviſum dat $T = -\frac{1}{2} - \frac{1}{2}z - \frac{1}{2}zz + \frac{1}{2}z^3$; ideoque $B = \frac{T}{S} = \frac{-1 - z - zz + z^3}{2 + 2z^4}$, poſito $z = 1$; ita ut ſit $B = \frac{-1}{2}$; fractiones ergo partiales quæſitæ ſunt $\frac{1}{2(1-z)^2} - \frac{1}{2(1-z)}$.

44. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat* $(p - qz)^3$ *ſequenti modo fractiones partiales ex hoc Factore oriundæ* $\frac{A}{(p-qz)^3} + \frac{B}{(p-qz)^2} + \frac{C}{p-qz}$ *reperientur.*

Ponatur $N=(p-qz)^3S$, ſitque fractio ex Factore S orta $=\frac{P}{S}$, erit $P=\frac{M-AS-B(p-qz)S-C(p-qz)^2S}{(p-qz)^3}$ $=$ Functioni integræ. Numerator ergo $M-AS-B(p-qz)\,S-C\,(p-qz)^2\,S$ ante omnia diviſibilis eſſe debet per $(p-qz)^3$; unde is, poſito $p-qz=0$, ſeu $z=\frac{p}{q}$, evaneſcere debet, eritque adeo $M-AS=0$, ideoque $A=\frac{M}{S}$, poſito $z=\frac{p}{q}$. Invento hoc pacto A, erit $M-AS$ diviſibile per $p-qz$; ponatur ergo $\frac{M-AS}{p-qz}=T$, atque $T-BS-C\,(p-qz)\,S$ adhuc per $(p-qz)^2$ erit diviſibile; fiet ergo $=0$, poſito $p-qz=0$; ex quo prodit $B=\frac{T}{S}$ poſito $z=\frac{p}{q}$. Sic autem invento B erit $T-BS$ diviſibile per $p-qz$. Hanc ob rem, poſito $\frac{T-BS}{p-qz}=V$, ſupereſt ut $V-CS$ diviſibile ſit per $p-qz$; eritque ergo $V-CS=0$, poſito $p-qz=0$, atque $C=\frac{V}{S}$, poſito $z=\frac{p}{q}$. Inventis ergo hoc modo numeratoribus A, B, C, fractiones partiales ex denominatoris N Factore $(p-qz)^3$ ortæ erunt $\frac{A}{(1-z)^3}+\frac{B}{(p-qz)^2}+\frac{C}{p-qz}$.

Exemplum.

Sit propoſita hæc fracta Functio $\frac{zz}{(1-z)^3(1+zz)}$, ex cujus denominatoris Factore cubico $(1-z)^3$ oriantur hæ fractiones partiales: $\frac{A}{(1-z)^3}+\frac{B}{(1-z)^2}+\frac{C}{1-z}$. Erit ergo $M=zz$ & $S=1+zz$; unde fit primum $A=\frac{zz}{1+zz}$.

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LIB. I. poſito $1 - z = 0$ ſeu $z = 1$; ex quo prodit $A = \frac{1}{2}$. Jam ponatur $T = \frac{M - AS}{1 - z}$, erit $T = \frac{\frac{1}{2}zz - \frac{1}{2}}{1 - z} = -\frac{1}{2} - \frac{1}{2}z$; unde oritur $B = \frac{-\frac{1}{2} - \frac{1}{2}z}{1 + zz}$, poſito $z = 1$, ita ut ſit $B = -\frac{1}{2}$. Ponatur porro $V = \frac{T - BS}{1 - z} = \frac{T + \frac{1}{2}S}{1 - z}$; erit $V = \frac{-\frac{1}{2}z + \frac{1}{2}zz}{1 - z} = -\frac{1}{2}z$; unde fit $C = \frac{V}{S} = \frac{-\frac{1}{2}z}{1 + zz}$ poſito $z = 1$, ita ut ſit $C = -\frac{1}{4}$. Quo circa fractiones partiales ex denominatoris Factore $(1 - z)^3$ ortæ erunt $\frac{1}{2(1-z)^3} - \frac{1}{2(1-z)^2} - \frac{1}{4(1-z)}$.

45. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat* $(p - qz)^n$; *fractiones partiales hinc ortæ* $\frac{A}{(p-qz)^n} + \frac{B}{(p-qz)^{n-1}} + \frac{C}{(p-qz)^{n-2}} + \ldots\ldots + \frac{K}{p-qz}$ *ſequenti modo invenientur.*

Ponatur denominator $N = (p - qz)^n Z$, atque, ratiocinium ut ante inſtituendo, reperietur ut ſequitur:

Primo $A = \frac{M}{Z}$, poſito $z = \frac{p}{q}$. Ponatur $P = \frac{M - AZ}{p - qz}$
Secundo $B = \frac{P}{Z}$, poſito $z = \frac{p}{q}$. Ponatur $Q = \frac{P - BZ}{p - qz}$
Tertio $C = \frac{Q}{Z}$, poſito $z = \frac{p}{q}$. Ponatur $R = \frac{Q - CZ}{p - qz}$
Quarto $D = \frac{R}{Z}$, poſito $z = \frac{p}{q}$. Ponatur $S = \frac{R - DZ}{p - qz}$
Quinto $E = \frac{S}{Z}$, poſito $z = \frac{p}{q}$. &c.

Hoc ergo modo ſi definiantur ſinguli numeratores conſtan-

tes

tes A, B, C, D, &c. invenientur omnes fractiones partiales, quæ ex denominatoris N Factore $(p - qz)^n$ naſcuntur.

EXEMPLUM.

Sit propoſita iſta Functio fracta $\frac{1+zz}{z^5(1+z^3)}$ ex cujus denominatoris Factore z^5 naſcantur hæ fractiones partiales $\frac{A}{z^5} + \frac{B}{z^4} + \frac{C}{z^3} + \frac{D}{z^2} + \frac{E}{z}$. Ad quarum numeratores conſtantes inveniendos, erit $M = 1 + zz$ atque $Z = 1 + z^3$; & $\frac{p}{q} = 0$. Sequens ergo calculus ineatur.

Primum eſt $A = \frac{M}{Z} = \frac{1+zz}{1+z^3}$, poſito $z = 0$; ergo $A = 1$.

Ponatur $P = \frac{M - AZ}{z} = \frac{zz - z^3}{z} = z - zz$. Eritque ſecundo $B = \frac{P}{Z} = \frac{z - zz}{1+z^3}$, poſito $z = 0$; ergo $B = 0$.

Ponatur $Q = \frac{P - BZ}{z} = \frac{z - zz}{z} = 1 - z$; eritque tertio $C = \frac{Q}{Z} = \frac{1-z}{1+z^3}$, poſito $z = 0$: ergo $C = 1$.

Ponatur $R = \frac{Q - CZ}{z} = \frac{-z - z^3}{z} = -1 - zz$; erit quarto $D = \frac{R}{Z} = \frac{-1 - zz}{1+z^3}$, poſito $z = 0$; ex quo fit $D = -1$.

Ponatur $S = \frac{R - DZ}{z} = \frac{-zz + z^3}{z} = -z + zz$; erit quinto $E = \frac{S}{Z} = \frac{-z + zz}{1+z^3}$, poſito $z = 0$; unde fit $E = 0$.

Quo circa fractiones partiales quæſitæ erunt hæ:
$\frac{1}{z^5} + \frac{0}{z^4} + \frac{1}{z^3} - \frac{1}{z^2} + \frac{0}{z}$.

LIB. I. 46. *Quæcunque ergo propoſita fuerit Functio rationalis fracta* $\frac{M}{N}$, *ea ſequenti modo in partes reſolvetur, atque in formam ſimpliciſſimam tranſmutabitur.*

Quærantur denominatoris N omnes Factores ſimplices ſive reales ſive imaginarii; quorum qui ſibi pares non habeant, ſeorſim tractentur & ex unoquoque per §. 41, fractio partialis eruatur. Quod ſi idem Factor ſimplex bis vel pluries occurrat, ii conjunctim ſumantur atque ex eorum producto, quod erit poteſtas formæ $(p - qz)^n$, quærantur fractiones partiales convenientes per §. 45. Hocque modo cum ex ſingulis Factoribus ſimplicibus denominatoris crutæ fuerint fractiones partiales, tum harum omnium aggregatum æquabitur Functioni propoſitæ $\frac{M}{N}$, niſi fuerit ſpuria; ſi enim fuerit ſpuria, pars integra inſuper extrahi atque ad iſtas fractiones partiales inventas adjici debebit, quo prodeat valor Functionis $\frac{M}{N}$ in forma ſimpliciſſima expreſſus. Perinde autem eſt ſive fractiones partiales ante extractionem partis integræ, ſive poſt quærantur. Eædem enim ex ſingulis denominatoris N Factoribus prodeunt fractiones partiales, ſive adhibeatur ipſe numerator M, ſive idem quocunque denominatoris N multiplo vel auctus vel minutus; id quod regulas datas contemplanti facile patebit.

EXEMPLUM.

Quæratur valor Functionis $\frac{1}{z^3(1-z)^2(1+z)}$ in forma ſimpliciſſima expreſſus. Sumatur primum Factor denominatoris ſolitarius $1+z$, qui dat $\frac{p}{q} = -1$, erit $M = 1$ & $Z = z^3 - 2z^4 + z^5$. Hinc ad fractionem $\frac{A}{1+z}$ inveniendam, erit $A = \frac{1}{z^3 - 2z^4 + z^5}$, poſito $z = -1$; ideoque fit $A =$

$-\frac{1}{4}$, atque ex Factore $1+z$ oritur hæc fractio partialis $\frac{-1}{4(1+z)}$. Jam ſumatur Factor quadratus $(1-z)^2$ qui dat $\frac{p}{q}=1$, $M=1$, & $Z=z^3+z^4$. Poſitis ergo fractionibus partialibus hinc ortis $\frac{A}{(1-z)^2}+\frac{B}{1-z}$, erit $A=\frac{1}{z^3+z^4}$, poſito $z=1$; ergo $A=\frac{1}{2}$; fiat $P=\frac{M-\frac{1}{2}Z}{1-z}$ $=\frac{1-\frac{1}{2}z^3-\frac{1}{2}z^4}{1-z}=1+z+zz+\frac{1}{2}z^3$; eritque $B=\frac{P}{Z}=\frac{1+z+zz+\frac{1}{2}z^3}{z^3+z^4}$, poſito $z=1$; ergo $B=\frac{7}{4}$ & fractiones partiales quæſitæ $\frac{1}{2(1-z)^2}+\frac{7}{4(1-z)}$. Denique tertius Factor cubicus z^3 dat $\frac{p}{q}=0$; $M=1$; & $Z=1-z-zz+z^3$. Poſitis ergo fractionibus partialibus his $\frac{A}{z^3}+\frac{B}{z^2}+\frac{C}{z}$; erit primum $A=\frac{M}{N}=\frac{1}{1-z-zz+z^3}$, poſito $z=0$; ergo $A=1$. Ponatur $P=\frac{M-Z}{z}=1+z-zz$, erit $B=\frac{P}{Z}$, poſito $z=0$; ergo $B=1$. Ponatur $Q=\frac{P-Z}{z}=2-zz$; erit $C=\frac{Q}{Z}$, poſito $z=0$; ergo $C=2$. Hanc ob rem Functio propoſita $\frac{1}{z^3(1-z)^2(1+z)}$ in hanc formam reſolvitur $\frac{1}{z^3}+\frac{1}{z^2}+\frac{2}{z}+\frac{1}{2(1-z)^2}+\frac{7}{4(1-z)}-\frac{1}{4(1+z)}$: nulla enim pars integra inſuper accedit, quia fractio propoſita non eſt ſpuria.

CAP. II.

CAPUT III.

De transformatione Functionum per ſubſtitutionem.

46 bis. *Si fuerit* y *Functio quæcunque ipſius* z, *atque* z *definiatur per novam variabilem* x, *tum quoque* y *per* x *definiri poterit.*

Cum ergo antea y fuiſſet Functio ipſius z, nunc nova quantitas variabilis x inducitur, per quam utraque priorum y & z definiatur. Sic, ſi fuerit $y = \frac{1 - zz}{1 + zz}$, atque ponatur $z = \frac{1 - x}{1 + x}$; hoc valore loco z ſubſtituto, erit $y = \frac{2x}{1 + xx}$. Sumpto ergo pro x valore quocunque determinato, ex eo reperientur valores determinati pro z & y, ſicque invenitur valor ipſius y reſpondens illi valori ipſius z qui ſimul prodiit. Uti ſi ſit $x = \frac{1}{2}$, fiet $z = \frac{1}{3}$, & $y = \frac{4}{5}$; reperitur autem quoque $y = \frac{4}{5}$, ſi in $\frac{1 - zz}{1 + zz}$, cui expreſſioni y æquatur, ponatur $z = \frac{1}{3}$.

Adhibetur autem hæc novæ variabilis introductio ad duplicem finem: vel enim hoc modo irrationalitas, qua expeſſio ipſius y per z data laborat, tollitur; vel quando ob æquationem altioris gradus, qua relatio inter y & z exprimitur, non licet Functionem explicitam ipſius z ipſi y æqualem exhibere, nova variabilis x introducitur, ex qua utraque y & z commode definiri queat: unde inſignis ſubſtitutionum uſus jam ſatis elucet, ex ſequentibus vero multo clarius perſpicietur.

47. *Si fuerit* $y = \sqrt{(a + bz)}$; *nova variabilis* x *per quam utraque* z & y *rationaliter exprimatur, ſequenti modo invenietur.*

Quoniam tam z quam y debet eſſe Functio rationalis ipſius x; perſpicuum eſt hoc obtineri ſi ponatur $\sqrt{(a + bz)} = bx$: Fiet enim primo $y = bx$; & $a + bz = bbxx$; hincque $z = bxx - \frac{a}{b}$. Quocirca utraque quantitas y & z per Functionem rationalem ipſius x exprimitur; ſcilicet cum ſit $y = \sqrt{(a + bz)}$ fiat $z = bxx - \frac{a}{b}$; erit $y = bx$.

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48. *Si fuerit* $y = (a + bz)^{m:n}$ *; nova variabilis* x *, per quam tam* y *quam* z *rationaliter exprimatur , ſic reperietur.* CAP. III.

Ponatur $y = x^m$, fietque $(a + bz)^{m:n} = x^m$ ideoque $(a + bz)^{1:n} = x$: ergo $a + bz = x^n$ & $z = \frac{x^n - a}{b}$. Sic ergo utraque quantitas y & z rationaliter per x definietur, ope ſcilicet ſubſtitutionis $z = \frac{x^n - a}{b}$, quæ præbet $y = x^m$. Quamvis igitur neque y per z, neque viciſſim z per y rationaliter exprimi poſſit; tamen utraque reddita eſt Functio rationalis novæ quantitatis variabilis x per ſubſtitutionem introductæ, ſcopo ſubſtitutionis omnino convenienter.

49. *Si fuerit* $y = (\frac{a + bz}{f + gz})^{m:n}$ *; requiritur nova quantitas variabilis* x *per quam utraque* y *&* z *rationaliter exprimatur.*

Manifeſtum primo eſt ſi ponatur $y = x^m$, quæſito ſatisfieri; erit enim $(\frac{a + bz}{f + gz})^{m:n} = x^m$, ideoque $\frac{a + bz}{f + gz} = x^n$; ex qua æquatione elicitur $z = \frac{a - fx^n}{gx^n - b}$; quæ ſubſtitutio præbet $y = x^m$. Hinc quoque intelligitur ſi fuerit $(\frac{\alpha + \beta y}{\gamma + \delta y})^n = (\frac{a + bz}{f + gz})^m$, tam y quam z rationaliter per x expreſſum iri, ſi utraque formula ponatur $= x^{mn}$; reperietur enim $y = \frac{\alpha - \gamma x^m}{\delta x^m - \beta}$ & $z = \frac{a - fx^n}{gx^n - b}$: qui caſus nil habent difficultatis.

50. *Si fuerit* $y = \sqrt{((a + bz)(c + dz))}$ *; ſubſtitutio idonea invenietur , qua* y *&* z *rationaliter exprimantur , hoc modo.*

Ponatur $\sqrt{((a + bz)(c + dz))} = (a + bz)x$, facile enim perſpicitur hinc valorem rationalem pro z eſſe prodituram; quia valor ipſius z per æquationem ſimplicem determinatur. Erit

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LIB. I. ergo $c + dz = (a + bz)\,xx$, hincque $z = \frac{c - axx}{bxx - d}$. Quare porro fiet $a + bz = \frac{bc - ad}{bxx - d}$; & ob $y = \sqrt{((a + bz)(c + dz))} = (a + bz)\,x$ habebitur $y = \frac{(bc - ad)\,x}{bxx - d}$. Functio ergo irrationalis $y = \sqrt{((a + bz)(c + dz))}$ ad rationalitatem perducitur ope ſubſtitutionis $z = \frac{c - axx}{bxx - d}$, quippe quæ dabit $y = \frac{(bc - ad)\,x}{bxx - d}$. Sic, ſi fuerit $y = \sqrt{(aa - zz)} = \sqrt{((a + z)(a - z))}$; ob $b = +1$, $c = a$, $d = -1$; ponatur $z = \frac{a - axx}{1 + xx}$, eritque $y = \frac{2ax}{1 + xx}$. Quoties ergo quantitas poſt ſignum $\sqrt{}$ habuerit duos Factores ſimplices reales, hoc modo reductio ad rationalitatem abſolvetur; ſin autem Factores bini ſimplices fuerint imaginarii, ſequenti modo uti præſtabit.

51. *Sit* $y = \sqrt{(p + qz + rzz)}$; *atque requiritur ſubſtitutio idonea pro* z *facienda, ut valor ipſius* y *ſiat rationalis.*

Pluribus modis hoc fieri poteſt, prout p & q fuerint quantitates vel affirmativæ vel negativæ. Sit primo p quantitas affirmativa, ac ponatur aa pro p; etiamſi enim p non ſit quadratum, tamen irrationalitas quantitatum conſtantium præſens negotium non turbat. Sit igitur

I. $y = \sqrt{(aa + bz + czz)}$; ac ponatur $\sqrt{(aa + bz + czz)} = a + xz$; erit $b + cz = 2ax + xxz$; unde fit $z = \frac{b - 2ax}{xx - c}$: tum vero erit $y = a + xz = \frac{bx - axx - ac}{xx - c}$; ubi z & y ſunt Functiones rationales ipſius x. Sit jam

II. $y = \sqrt{(aazz + bz + c)}$; ac ponatur $\sqrt{(aazz + bz + c)} = az + x$; erit $bz + c = 2axz + xx$, & $z = \frac{xx - c}{b - 2ax}$. Tum autem fit $y = az + x = \frac{-ac + bx - axx}{b - 2ax}$.

III. Si fuerint p & r quantitates negativæ; tum, niſi ſit $qq > 4pr$, valor ipſius y ſemper erit imaginarius. Quod ſi autem fuerit $qq > 4pr$; expreſſio $p + qz + rzz$ in duos Factores

reſolvi poterit, qui caſus ad §. præced. reducitur. Sæpenumero autem commodius ad hanc formam reducitur, $y = \sqrt{(aa + (b + cz)(d + ez))}$; pro qua ad rationalitatem perducenda ponatur $y = a + (b + cz)x$, eritque $d + ez = 2ax + bxx + cxxz$; unde fit $z = \frac{d - 2ax - bxx}{cxx - e}$; & $y = \frac{-ae + (cd - be)x - acxx}{cxx - e}$. Interdum commodius fieri poteſt reductio ad hanc formam, $y = \sqrt{(aazz + (b + cz)(d + ez))}$. Tum ponatur $y = az + (b + cz)x$; erit $d + ez = 2axz + bxx + cxxz$ & $z = \frac{bxx - d}{e - 2ax - cxx}$, atque $y = \frac{-ad + (be - cd)x - abxx}{e - 2ax - cxx}$. CAP. III.

EXEMPLUM.

Si habeatur iſta ipſius z Functio irrationalis $y = \sqrt{(-1 + 3z - zz)}$; quæ cum reduci queat ad hanc formam $y = \sqrt{(1 - 2 + 3z - zz)} = \sqrt{(1 - (1 - z)(2 - z))}$; ponatur $y = 1 - (1 - z)x$, erit $-2 + z = -2x + xx - xxz$ & $z = \frac{2 - 2x + xx}{1 + xx}$. Deinde eſt $1 - z = \frac{-1 + 2x}{1 + xx}$ & $y = 1 - (1 - z)x = \frac{1 + x - xx}{1 + xx}$. Atque hi ſunt fere caſus, quos Algebra indeterminata, ſeu methodus *Diophantæa*, ſuppeditat; neque alios caſus in his tractatis non comprehenſos per ſubſtitutionem rationalem ad rationalitatem reducere licet. Quocirca ad alterum ſubſtitutionis uſum monſtrandum progredior.

52. *Si* y *ejuſmodi fuerit Functio ipſius* z *ut ſit* $ay^\alpha + bz^\beta + cy^\gamma z^\delta = 0$, *invenire novam variabilem* x, *per quam valores ipſarum* y *&* z *explicite aſſignari queant.*

Quoniam reſolutio æquationum generalis non habetur, ex æquatione propoſita $ay^\alpha + bz^\beta + cy^\gamma z^\delta = 0$ neque y per z

LIB. I. neque viciſſim z per y exhiberi poteſt. Quo igitur huic incommodo remedium afferatur; ponatur $y = x^m z^n$, eritque $ax^{\alpha m} z^{\alpha n} + b z^{\beta} + c x^{\gamma m} z^{\gamma n + \delta} = 0$. Determinetur nunc exponens n ita ut ex hac æquatione valor ipſius z definiri queat: quod tribus modis præſtari poteſt.

I. Sit $\alpha n = \beta$; ideoque $n = \frac{\beta}{\alpha}$; erit, æquatione per $z^{\alpha n} = z^{\beta}$ diviſa, $ax^{\alpha m} + b + cx^{\gamma m} z^{\gamma n - \beta + \delta} = 0$; unde oritur $z = \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{1}{\gamma n - \beta + \delta}}$, ſive

$$z = \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{\alpha}{\beta\gamma - \alpha\beta + \alpha\delta}}, \ \&$$

$$y = x^m \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{\beta}{\beta\gamma - \alpha\beta + \alpha\delta}}.$$

II. Sit $\beta = \gamma n + \delta$ ſeu $n = \frac{\beta - \delta}{\gamma}$; erit, æquatione per z^{β} diviſa, $ax^{\alpha m} z^{\alpha n - \beta} + b + cx^{\gamma m} = 0$; unde oritur

$$z = \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{1}{\alpha n - \beta}} = \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{\gamma}{\alpha\beta - \alpha\delta - \beta\gamma}};$$

atque $y = x^m \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{\beta - \delta}{\alpha\beta - \alpha\delta - \beta\gamma}}$.

III. Sit $\alpha n = \gamma n + \delta$, ſeu $n = \frac{\delta}{\alpha - \gamma}$; erit, æquatione per $z^{\alpha n}$ diviſa, $ax^{\alpha m} + b z^{\beta - \alpha n} + cx^{\gamma m} = 0$; unde oritur

$$z = \left(\frac{-ax^{\alpha m} - cx^{\gamma m}}{b}\right)^{\frac{1}{\beta - \alpha n}} =$$

$$(\frac{-a x^{\alpha m} - c x^{\gamma m}}{b})^{\frac{\alpha - \gamma}{\alpha\beta - \beta\gamma - \alpha\delta}};\ \text{atque}$$

$$y = x^m (\frac{-a x^{\alpha m} - c x^{\gamma m}}{b})^{\frac{\delta}{\alpha\beta - \beta\gamma - \alpha\delta}}.$$

Tribus igitur diverſis modis erutæ ſunt Functiones ipſius x, quæ ipſis z & y ſunt æquales. Præterea vero pro m numerum pro lubitu ſubſtituere licet cyphra excepta; ſicque formulæ ad commodiſſimam expreſſionem reduci poterunt.

EXEMPLUM.

Exprimatur natura Functionis y per hanc æquationem $y^3 + z^3 - cyz = 0$; atque quærantur Functiones ipſius x ipſis y & z æquales. Erit ergo $a = -1$; $b = -1$; $\alpha = 3$; $\beta = 3$; $\gamma = 1$; & $\delta = 1$. Hinc primus modus dabit, poſito $m = 1$, $z = (\frac{x^3+1}{cx})^{-1}$ & $y = x(\frac{x^3+1}{cx})^{-1}$, ſive $z = \frac{cx}{1+x^3}$ & $y = \frac{cxx}{1+x^3}$; quarum expreſſionum utraque adeo eſt rationalis.

Secundus modus vero dabit hos valores:

$$z = (\frac{cx-1}{x^3})^{1:3},\ \&\ y = x(\frac{cx-1}{x^3})^{2:3},\ \text{ſive}$$

$$z = \frac{1}{x}\sqrt[3]{(cx-1)},\ \&\ y = \frac{1}{x}\sqrt[3]{(cx-1)^2}.$$

Tertius modus ita rem expediet ut ſit

$$z = (cx - x^3)^{2:3},\ \&\ y = x\,(cx - x^3)^{1:3}.$$

53. *Hinc à poſteriori intelligitur cujuſmodi æquationes, quibus valor Functionis* y *per* z *determinatur, hoc modo novam variabilem* x *introducendo reſolvi queant.*

Ponamus enim reſolutione jam inſtituta prodiiſſe has deter-

LIB. I. minationes $z = \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{p:r}$, atque $y = x \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{q:r}$; eritque $y^p = x^p z^q$; & hinc $x = yz^{-q:p}$. Cum igitur ſit $z^{r:p} = \frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}$, ſi loco x ejus valorem $yz^{-q:p}$ ſubſtituamus; prodibit iſta æquatio $z^{r:p} = \frac{ay^\alpha z^{-\alpha q:p} + by^\beta z^{-\beta q:p} + cy^\gamma z^{-\gamma q:p} + \&c.}{A + By^\mu z^{-\mu q:p} + Cy^\nu z^{-\nu q:p} + \&c.}$; quæ reducitur ad hanc $Az^{r:p} + By^\mu z^{(r-\mu q):p} + Cy^\nu z^{(r-\nu q):q} + \&c. = ay^\alpha z^{-\alpha q:p} + by^\beta z^{-\beta q:p} + cy^\gamma z^{-\gamma q:p} + \&c.$ quæ multiplicata per $z^{\alpha q:p}$ tranſibit in hanc: $Az^{(\alpha q+r):p} + By^\mu z^{(\alpha q-\mu q+r):p} + Cy^\nu z^{(\alpha q-\nu q+r):p} + \&c. = ay^\alpha + by^\beta z^{(\alpha q-\beta q):p} + cy^\gamma z^{(\alpha q-\gamma q):p} + \&c.$ Ponatur $\frac{\alpha q + r}{p} = m$ & $\frac{\alpha q - \beta q}{p} = n$: fiet [*] $p = \alpha - \beta$; $q = n$, & $r = \alpha m - \beta m - \alpha n$: atque naſcetur iſta æquatio: $Az^m + By^\mu z^{m-\mu n:(\alpha-\beta)} + Cy^\nu z^{m-\nu n:(\alpha-\beta)} + \&c. = ay^\alpha + by^\beta z^n + cy^\gamma z^{(\alpha-\gamma)n:(\alpha-\beta)} + \&c.$ quæ propterea ita reſolvetur ut ſit:

$$z = \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{\frac{\alpha-\beta}{\alpha m - \beta m - \alpha n}} \;\&$$

$$y = x\left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{\frac{n}{\alpha m - \beta m - \alpha n}}$$

Vel ponatur $\frac{\alpha q + r}{p} = m$, & $\frac{\alpha q - \mu q + r}{p} = n$, erit $m - n$

[*] Quoniam tres adſunt indeterminatæ, & duæ tantum æquationes, licuit ponere $p = \alpha - \beta$.

$= \frac{\mu q}{p}$; & $\frac{q}{p} = \frac{m - n}{\mu}$, atque $\frac{r}{p} = m - \frac{\alpha m + \alpha n}{\mu}$. Hinc fit [*] $p = \mu$; $q = m - n$; & $r = \mu m - \alpha m + \alpha n$; atque hæc æquatio reſultabit : Cap. III.

$$Az^m + By^\mu z^n + Cy^\nu z^{m - \nu(m-n):\mu} + \&c. = ay^\alpha + by^\beta z^{(\alpha - \beta)(m-n):\mu} + cy^\gamma z^{(\alpha-\gamma)(m-n):\mu}$$

+ &c. quæ ita reſolvetur ut ſit :

$$z = \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{\frac{\mu}{\mu m - \alpha m + \alpha n}} \;\&$$

$$y = x\left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.}\right)^{\frac{m-n}{\mu m - \alpha m + \alpha n}}.$$

54. *Si* y *ita pendeat a* z *ut ſit* $ayy + byz + czz + dy + ez = 0$, *ſequenti modo tam* y *quam* z *rationaliter per novam variabilem* x *exprimetur.*

Ponatur $y = xz$, erit diviſione per z facta : $axxz + bxz + cz + dx + e = 0$, ex qua reperitur $z = \frac{-dx - e}{axx + bx + c}$, & $y = \frac{-dxx - ex}{axx + bx + c}$.

At vero ad formam propoſitam reduci poteſt hæc æquatio inter y & z: $ayy + byz + czz + dy + ez + f = 0$ diminuendo vel augendo utramque variabilem certa quadam quantitate conſtante, unde & hæc æquatio per novam variabilem x rationaliter explicari poteſt.

55. *Si* y *ita pendeat a* z, *ut ſit* $ay^3 + by^2z + cyz^2 + dz^3 + eyy + fyz + gzz = 0$; *ſequenti modo tam* y *quam* z *rationaliter per novam variabilem* x *exprimi poterit.*

Ponatur $y = xz$, & facta ſubſtitutione tota æquatio per zz dividi poterit : prodibit autem $ax^3z + bxxz + cxz + dz + exx + fx + g = 0$. Unde oritur $z = \frac{-exx - fx - g}{ax^3 + bxx + cx + d}$ ex quo erit $y = \frac{-ex^3 - fxx - gx}{ax^3 + bxx + cx + d}$.

[*] Eadem cauſa qua ſupra fuit $p = \alpha - \beta$.

Ex his caſibus facile intelligitur quemadmodum æquationes altiorum graduum, quibus y per z definitur, comparatæ eſſe debeant, ut hujuſmodi reſolutio locum habere queat. Ceterum hi caſus in ſuperioribus formulis §. 53. continentur: at, quia formulæ generales non tam facile ad hujuſmodi caſus ſæpius occurrentes accommodantur, viſum eſt horum aliquos ſeorſim evolvere.

56. *Si* y *ita pendeat a* z *ut ſit* $ayy + byz + czz = d$ *hoc modo utraque quantitas* y *&* z *per novam variabilem* x *exprimetur.*

Ponatur $y = xz$, eritque $(axx + bx + c)zz = d$, ideoque $z = \sqrt{\frac{d}{axx + bx + c}}$ & $y = x\sqrt{\frac{d}{axx + bx + c}}$.

Simili modo ſi fuerit, $ay^3 + by^2z + cyz^2 + dz^3 = ey + fz$; poſito $y = xz$, tota æquatio per z diviſa dabit $(ax^3 + bxx + cx + d)zz = ex + f$; unde oritur $z = \sqrt{\frac{ex + f}{ax^3 + bxx + cx + d}}$; & $y = x\sqrt{\frac{ex + f}{ax^3 + bxx + cx + d}}$. Hi autem caſus aliique ſimiles reſolutiones admittentes comprehenduntur in ſequente paragrapho.

57. *Si* y *ita pendeat a* z *ut ſit* $ay^m + by^{m-1}z + cy^{m-2}z^2 + dy^{m-3}z^3 + \&c. = \alpha y^n + \beta y^{n-1}z + \gamma y^{n-2}z^2 + \delta y^{n-3}z^3 + \&c.$, *ſequenti modo tam* z *quam* y *commode per novam variabilem* x *exprimetur.*

Sit $y = xz$, atque facta ſubſtitutione tota æquatio dividi poterit per z^n, ſiquidem exponens m ſit major quam n; eritque $(ax^m + bx^{m-1} + cx^{m-2} + \&c.)z^{m-n} = \alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \&c.$ unde obtinebitur

$$z = \left(\frac{\alpha x^{n} + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \&c.}{ax^{m} + bx^{m-1} + cx^{m-2} + dx^{m-3} + \&c.}\right)^{1:(m-n)} \&$$

$$y = x\left(\frac{\alpha x^{n} + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \&c.}{ax^{m} + bx^{m-1} + cx^{m-2} + dx^{m-3} + \&c.}\right)^{1:(m-n)}$$

Hæc ſcilicet reſolutio locum habet, ſi in æquatione naturam Functionis y per z exprimente, duplex tantum ubique occurrit dimenſionum ab y & z ſumptarum numerus; uti in caſu tractato in ſingulis terminis numerus dimenſionum vel eſt m vel n.

58. *Si in æquatione inter* y *&* z *triplicis generis dimenſiones occurrant, quarum ſumma tantum ſuperet mediam, quantum hæc media infimam, ope reſolutionis æquationis quadratæ variabiles* y *&* z *per novam* x *exprimi poterunt.*

Si enim ponatur $y = xz$, diviſione per minimam ipſius z poteſtatem facta, valor ipſius z per x, ope extractionis radicis quadratæ exhibebitur, id quod ex ſequentibus exemplis erit manifeſtum.

EXEMPLUM I.

Sit $ay^{3} + by^{2}z + cyzz + dz^{3} = 2eyy + 2fyz + 2gzz + hy + iz$; ac ponatur $y = xz$: erit, diviſione per z facta, $(ax^{3} + bxx + cx + d)zz = 2(exx + fx + g)z + hx + i$; ex qua ſequens ipſius z obtinebitur valor:

$$z = \frac{exx + fx + g \pm \sqrt{((exx + fx + g)^{2} + (ax^{3} + bxx + cx + d)(hx + i))}}{ax^{3} + bxx + cx + d},$$

quo invento erit $y = xz$.

EXEMPLUM II.

Sit $y^{5} = 2az^{3} + by + cz$; ac, poſito $y = xz$, erit $x^{5}z^{4} = 2azz + bx + c$; ex qua reperitur $zz = \frac{a \pm \sqrt{(aa + bx^{6} + cx^{5})}}{x^{5}}$; &

$$z = \frac{\pm\sqrt{(a \pm \sqrt{(aa + bx^{6} + cx^{5})})}}{xx\sqrt{x}} \;\&\; y = \frac{\pm\sqrt{(a \pm \sqrt{(aa + bx^{6} + cx^{5})})}}{x\sqrt{x}}.$$

EXEMPLUM III.

Sit $y^{10} = 2ayz^6 + byz^3 + cz^4$, in qua cum dimenſiones ſint 10, 7, & 4, ponatur $y = xz$; atque æquatio per z^4 diviſa abibit in hanc: $x^{10} z^6 = 2axz^3 + bx + c$ ſeu $z^6 = \frac{2axz^3 + bx + c}{x^{10}}$; unde invenitur $z^3 = \frac{ax \pm x\sqrt{(aa + bx^9 + cx^8)}}{x^{10}}$; ideoque erit $z = \frac{\sqrt[3]{(a \pm \sqrt{(aa + bx^9 + cx^8)})}}{x^3}$; atque $y = \frac{\sqrt[3]{(a \pm \sqrt{(aa + bx + cx^8)})}}{x^2}$. Ex quibus exemplis uſus hujuſmodi ſubſtitutionum abunde perſpicitur.

CAPUT IV.

De explicatione Functionum per ſeries infinitas.

59. CUM Functiones fractæ atque irrationales ipſius z non in forma integra $A + Bz + Cz^2 + Dz^3 +$ &c. continentur, ita ut terminorum numerus ſit finitus, quæri ſolent hujuſmodi expreſſiones in infinitum excurrentes, quæ valorem cujuſvis Functionis ſive fractæ ſive irrationalis exhibeant. Quin etiam natura Functionum tranſcendentium melius intelligi cenſetur, ſi per ejuſmodi formam, etſi infinitam, exprimantur. Cum enim natura Functionis integræ optime perſpiciatur, ſi ſecundum diverſas poteſtates ipſius z explicetur, atque adeo ad formam $A + Bz + Cz^2 + Dz^3 +$ &c. reducatur, ita eadem forma aptiſſima videtur ad reliquarum Functionum omnium indolem menti repræſentandam, etiamſi terminorum numerus ſit revera infinitus. Perſpicuum autem eſt nullam Functionem non integram ipſius z per numerum hujuſmodi terminorum $A + Bz + Cz^2 +$ &c. finitum exponi poſſe; eo ipſo enim

Functio foret integra; num vero per hujuſmodi terminorum ſeriem infinitam exhiberi poſſit, ſi quis dubitet, hoc dubium per ipſam evolutionem cujuſque Functionis tolletur. Quo autem hæc explicatio latius pateat, præter poteſtates ipſius z exponentes integros affirmativos habentes, admitti debent poteſtates quæcunque. Sic dubium erit nullum quin omnis Functio ipſius z in hujuſmodi expreſſionem infinitam transmutari poſſit: $Az^{\alpha} + Bz^{\beta} + Cz^{\gamma} + Dz^{\delta} +$ &c. denotantibus exponentibus α, β, γ, δ, &c. numeros quoſcunque.

60. *Per diviſionem autem continuam intelligitur fractionem* $\frac{a}{\alpha + \beta z}$ *reſolvi in hanc ſeriem infinitam* $\frac{a}{\alpha} - \frac{a\beta z}{\alpha^2} + \frac{a\beta^2 z^2}{\alpha^3} - \frac{a\beta^3 z^3}{\alpha^4} + \frac{a\beta^4 z^4}{\alpha^5} -$ *&c., quæ, cum quilibet terminus ad ſequentem habeat rationem conſtantem* $1 : \frac{\beta z}{\alpha}$, *vocatur ſeries geometrica.*

Poteſt vero quoque hæc ſeries ita inveniri, ut ipſa initio pro incognita habeatur: ponatur enim $\frac{a}{\alpha + \beta z} = A + Bz + Cz^2 + Dz^3 + Ez^4 +$ &c. atque ad æqualitatem producendam quærantur coëfficientes A, B, C, D, &c. Erit ergo $a = (\alpha + \beta z)(A + Bz + Cz^2 + Dz^3 +$ &c.$)$, & multiplicatione actu peracta fiet

$$\begin{aligned} a = \alpha A + \alpha Bz + \alpha Cz^2 + \alpha Dz^3 + \alpha Ez^4 + \&c. \\ + \beta Az + \beta Bz^2 + \beta Cz^3 + \beta Dz^4 + \&c. \end{aligned}$$

Quamobrem eſſe debet $a = \alpha A$, ideoque $A = \frac{a}{\alpha}$, & coëfficientium uniuſcujuſque poteſtatis ipſius z ſumma nihilo æqualis eſt ponenda: unde prodibunt hæ æquationes,

$\alpha B + \beta A = 0$ cognito ergo quovis coëfficiente
$\alpha C + \beta B = 0$ facile reperitur ſequens; ſi enim
$\alpha D + \beta C = 0$ fuerit coëfficiens termini cujuſque $= P$
$\alpha E + \beta D = 0$ & ſequens $= Q$ erit $\alpha Q + \beta P = 0$
&c. ſive $Q = -\frac{\beta P}{\alpha}$.

 Cum igitur terminus primus A ſit determinatus $= \frac{a}{\alpha}$ ex eo ſequentes litteræ B, C, D, &c. definiuntur eodem modo, quo ex diviſione ſunt orti. Ceterum ex inſpectione perſpicitur in ſerie infinita pro $\frac{a}{\alpha + \beta z}$ inventa poteſtatis z^n coëfficientem fore $= \pm \frac{a \beta^n}{\alpha^{n+1}}$, ubi ſignum $+$ locum habet ſi n ſit numerus par, ſignum $-$ autem ſi n ſit numerus impar: ſeu coëfficiens erit $= \frac{a}{\alpha} \left(\frac{-\beta}{\alpha}\right)^n$.

61. *Simili modo ope diviſionis continuatæ hæc Functio fracta* $\frac{a + bz}{\alpha + \beta z + \gamma zz}$ *in ſeriem infinitam converti poteſt.*

Cum autem diviſio ſit tædioſa, neque tam facile naturam ſeriei infinitæ oſtendat, commodius erit ſeriem quæſitam fingere, atque modo ante tradito determinare. Sit igitur

$$\frac{a + bz}{\alpha + \beta z + \gamma zz} = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$$

multiplicetur utrinque per $\alpha + \beta z + \gamma zz$, atque fiet

$$\begin{aligned} a + bz = \alpha A &+ \alpha Bz + \alpha Cz^2 + \alpha Dz^3 + \alpha Ez^4 + \&c. \\ &+ \beta Az + \beta Bz^2 + \beta Cz^3 + \beta Dz^4 + \&c. \\ &+ \gamma Az^2 + \gamma Bz^3 + \gamma Cz^4 + \&c. \end{aligned}$$

Hinc erit $\alpha A = a$; $\alpha B + \beta A = b$; unde reperitur $A = \frac{a}{\alpha}$ & $B = \frac{b}{\alpha} - \frac{a\beta}{\alpha\alpha}$; reliquæ vero litteræ ex ſequentibus æquationibus determinabuntur:

$\alpha C + \beta B + \gamma A = 0$
$\alpha D + \beta C + \gamma B = 0$
$\alpha E + \beta D + \gamma C = 0$
$\alpha F + \beta E + \gamma D = 0$
&c.

hinc ergo ex binis quibuſque coëfficientibus contiguis ſequens reperitur. Sic ſi duo coëfficientes contigui fuerint P, Q & ſequens R, erit $\alpha R + \beta Q + \gamma P = 0$ ſeu $R = \frac{-\beta Q - \gamma P}{\alpha}$.

Cum igitur duæ litteræ primæ A & B jam ſint inventæ ſequentes C, D, E, F &c. omnes ſucceſſive ex iis invenientur,

tur, ſicque reperietur Series infinita $A + Bz + Cz^2 + Dz^3 + \&c.$
Functioni fractæ propoſitæ $\frac{a + bz}{\alpha + \beta z + \gamma z^2}$ æqualis.

EXEMPLUM.

Si fuerit propoſita hæc fractio $\frac{1 + 2z}{1 - z - zz}$, huicque æqualis ſtatuatur Series $A + Bz + Cz^2 + Dz^3 + \&c.$ ob $a = 1$; $b = 2$; $\alpha = 1$; $\beta = -1$; $\gamma = -1$; erit $A = 1$; $B = 3$; tum vero erit

$C = B + A$ quilibet ergo coëfficiens æqualis eſt ſum-
$D = C + B$ mæ duorum præcedentium; quare ſi cog-
$E = D + C$ niti fuerint duo coëfficientes contigui
$F = E + D$ P & Q, erit ſequens $R = P + Q$.
&c.

Cum igitur duo coëfficientes primi A & B ſint cogniti, fractio propoſita $\frac{1 + 2z}{1 - z - zz}$ in hanc Seriem infinitam tranſmutatur $1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \&c.$, quæ nullo negotio quouſque libuerit continuari poteſt.

62. Ex his jam ſatis intelligitur indoles Serierum infinitarum, in quas Functiones fractæ tranſmutantur; tenent enim ejuſmodi legem, ut quilibet terminus ex aliquot præcedentibus determinari poſſit. Scilicet, ſi denominator fractionis propoſitæ fuerit $\alpha + \beta z$, atque Series infinita ſtatuatur

$$A + Bz + Cz^2 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + \&c.;$$

quilibet coëfficiens Q ex præcedente P ſolo ita definietur ut ſit $\alpha Q + \beta P = 0$. Sin denominator fuerit trinomium $\alpha + \beta z + \gamma zz$, quilibet Seriei coëfficiens R ex duobus præcedentibus Q & P ita definietur ut ſit $\alpha R + \beta Q + \gamma P = 0$: ſimili modo ſi denominator fuerit quadrinomium, ut $\alpha + \beta z + \gamma zz + \delta z^3$, quilibet ſeriei coëfficiens S ex tribus antecedentibus R, Q & P ita determinabitur, ut ſit $\alpha S + \beta R + \gamma Q + \delta P = 0$,

LIB. I. ſicque de ceteris. In his ergo Seriebus quilibet terminus determinatur ex aliquot antecedentibus ſecundum legem quandam conſtantem, quæ lex ex denominatore fractionis hanc Seriem producentis ſponte apparet. Vocari autem hæ Series a Celeb. MOIVRÆO, qui earum naturam maxime eſt ſcrutatus, ſolent *recurrentes*, propterea quod ad terminos antecedentes eſt recurrendum, ſi ſequentes inveſtigare velimus.

63. Ad harum porro Serierum formationem requiritur ut denominatoris terminus conſtans α non ſit $= 0$: cum enim inventus ſit terminus Seriei primus $A = \frac{a}{\alpha}$, tum is, tum omnes ſequentes fierent infiniti, ſi eſſet $\alpha = 0$. Hoc ergo caſu excluſo, quem deinceps evolvam, Functio fracta in Seriem infinitam recurrentem tranſmutanda, hujuſmodi habebit formam $\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z - \gamma z^3 - \delta z^4 - \&c.}$; ubi primum denominatoris terminum pono $= 1$, huc enim ſemper fractio reduci poteſt, niſi is ſit $= 0$; reliquos autem denominatoris terminos omnes tanquam negativos contemplor, ut Seriei hinc formatæ omnes termini fiant affirmativi. Quod ſi enim Series recurrens hinc orta ponatur $A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.$ coëfficientes ita determinabuntur ut ſit

$$A = a$$
$$B = \alpha A + b$$
$$C = \alpha B + \beta A + c$$
$$D = \alpha C + \beta B + \gamma A + d$$
$$E = \alpha D + \beta C + \gamma B + \delta A + e$$
$$\&c.$$

Quilibet ergo coëfficiens æqualis eſt aggregato ex multiplis aliquot præcedentium una cum numero quodam, quem numerator præbet. Niſi autem numerator in infinitum progrediatur, hæc additio mox ceſſabit, atque quivis terminus ſecundum legem conſtantem ex aliquot præcedentibus determinabitur. Ne ergo lex progreſſionis uſquam turbetur conveniet

Functionem fractam genuinam adhibere : ſi enim fractio ſpuria accipiatur , tum pars integra in ea contenta ad Seriem accedet , atque in illis terminis , quos vel auget vel minuit , legem progreſſionis interrumpet. Exempli gratia hæc fractio ſpuria $\frac{1 + 2z - z^3}{1 - z - zz}$, præbebit hanc Seriem $1 + 3z + 4zz + 6z^3 + 10z^4 + 16z^5 + 26z^6 + 42z^7 +$ &c. ubi a lege , qua quivis coëfficiens eſt ſumma duorum præcedentium , terminus quartus $6z^3$ excipitur.

64. Peculiarem contemplationem Series recurrentes merentur , ſi denominator fractionis , unde oriuntur , fuerit poteſtas. Sic , ſi iſta fractio $\frac{a + bz}{(1 - \alpha z)^2}$ in Seriem reſolvatur , prodit

$$\begin{array}{llllll} a & + 2\alpha a \, z & + 3\alpha^2 a \, z^2 & + 4\alpha^3 a \, z^3 & + 5\alpha^4 a \, z^4 & + \&c. \\ & + \; b & + 2\alpha b & + 3\alpha^2 b & + 4\alpha^3 b & \end{array}$$

in qua coëfficiens poteſtatis z^n erit $(n + 1)\alpha^n a + n\alpha^{n-1} b$. Erit tamen hæc Series recurrens , quia quilibet terminus ex duobus præcedentibus determinatur , cujus determinationis lex perſpicitur ex denominatore evoluto $1 - 2\alpha z + \alpha\alpha zz$. Si ponatur $\alpha = 1$ & $z = 1$, abit Series in progreſſionem arithmeticam generalem $a + (2a + b) + (3a + 2b) + (4a + 3b) +$ &c. cujus differentiæ ſunt conſtantes. Omnis ergo progreſſio arithmetica eſt Series recurrens : ſi enim ſit
$A + B + C + D + E + F +$ &c. progreſſio arithmetica, erit
$C = 2B - A$; $D = 2C - B$; $E = 2D - C$, &c.

65. Deinde hæc fractio $\frac{a + bz + czz}{(1 - \alpha z)^3}$ ob $\frac{1}{(1 - \alpha z)^3} = (1 - \alpha z)^{-3}$ $= 1 + 3\alpha z + 6\alpha^2 z^2 + 10\alpha^3 z^3 + 15\alpha^4 z^4 +$ &c. tranſmutabitur in hanc Seriem infinitam :

$$\begin{array}{llllll} a & + 3\alpha a \, z & + 6\alpha^2 a & + 10\alpha^3 a & + 15\alpha^4 a & + \&c. \\ & + \; b & + 3\alpha b \, z^2 & + 6\alpha^2 b \, z^3 & + 10\alpha^3 b \, z^4 & \\ & & + \; c & + 3\alpha c & + 6\alpha^2 c & \end{array}$$

LIB. I. in qua poteſtas z^n coëfficientem habebit $\frac{(n+1)(n+2)}{1.\ 2} \alpha^n a + \frac{n(n+1)}{1.\ 2} \alpha^{n-1} b + \frac{(n-1)n}{1.\ 2} \alpha^{n-2} c$. Quod ſi autem ponatur $\alpha = 1$ & $z = 1$, Series hæc abibit in progreſſionem generalem ſecundi ordinis, cujus differentiæ ſecundæ ſunt conſtantes. Deſignet $A + B + C + D + E +$ &c. hujuſmodi progreſſionem, erit ea ſimul Series recurrens, cujus quilibet terminus ex tribus antecedentibus ita determinatur ut ſit $D = 3C - 3B + A$; $E = 3D - 3C + B$; $F = 3E - 3D + C$ &c. Cum igitur terminorum in progreſſione arithmetica procedentium ſecundæ differentiæ quoque ſint æquales, nempe $= 0$, hæc proprietas quoque ad progreſſiones arithmeticas extenditur.

66. Simili modo hæc fractio $\frac{a + bz + czz + dz^3}{(1 - \alpha z)^4}$ dabit Seriem infinitam, in qua poteſtas ipſius z quæcunque z^n hunc habebit coëfficientem $\frac{(n+1)(n+2)(n+3)}{1.\ 2.\ 3} \alpha^n a + \frac{n(n+1)(n+2)}{1.\ 2.\ 3} \alpha^{n-1} b + \frac{(n-1)n(n+1)}{1.\ 2.\ 3} \alpha^{n-2} c + \frac{(n-2)(n-1)n}{1.\ 2.\ 3} \alpha^{n-3} d$: poſito ergo $\alpha = 1$ & $z = 1$, hæc Series in ſe complectetur omnes progreſſiones algebraicas tertii ordinis, quarum differentiæ tertiæ ſunt conſtantes: omnes ergo hujus ordinis progreſſiones, cujuſmodi ſit $A+B+C+D+E+F+$ &c. erunt ſimul recurrentes ex denominatore $1 - 4z + 6zz - 4z^3 + z^4$ ortæ; unde erit $E = 4D - 6C + 4B - A$; $F = 4E - 6D + 4C - B$ &c., quæ proprietas ſimul in omnes progreſſiones inferiorum ordinum competit.

67. Hoc modo oſtendetur omnes progreſſiones algebraicas cujuſcunque ordinis, quæ tandem ad differentias conſtantes deducunt, eſſe Series recurrentes, quarum lex definiatur ex denominatore $(1 - z)^n$, exiſtente n numero majore quam is, qui ordinem progreſſionis indicat. Cum igitur $a^m + (a+b)^m +$

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$(a+2b)^m + (a+3b)^m +$ &c. exhibeat progreſſionem ordinis m; erit ob naturam Serierum recurrentium

$$0 = a^m - \frac{n}{1}(a+b)^m + \frac{n(n-1)}{1.\ 2}(a+2b)^m - \frac{n(n-1)(n-2)}{1.\ 2.\ 3}(a+3b)^m + \ldots \pm \frac{n}{1}(a+(n-1)b)^m \mp (a+nb)^m;$$

ubi ſigna ſuperiora valent ſi n ſit numerus par, inferiora autem ſi n ſit numerus impar. Hæc ergo æquatio ſemper eſt vera ſi fuerit n numerus integer major quam m. Hinc ergo intelligitur quam late pateat doctrina de Seriebus recurrentibus.

68. Si denominator fuerit poteſtas non binomii ſed multinomii, natura Seriei quoque alio modo explicari poteſt. Sit nempe hæc fractio $\frac{1}{(1-\alpha z-\beta z^2-\gamma z^3-\delta z^4-\&c.)^{m+1}}$ propoſita, erit Series infinita hinc nata

$$1+\frac{(m+1)}{1}\alpha z+\left\{\begin{matrix}\frac{(m+1)(m+2)}{1.\ 2}\alpha^2\\ +\frac{(m+1)}{1}\beta\end{matrix}\right\}z^2+\left\{\begin{matrix}\frac{(m+1)(m+2)(m+3)}{1.\ 2.\ 3}\alpha^3\\ +\frac{(m+1)(m+2)}{1.\ 2}2\alpha\beta\\ +\frac{(m+1)}{1}\gamma\end{matrix}\right\}z^3+\&c.$$

Ad naturam hujus Seriei penitus inſpiciendam, exponatur hæc Series per litteras generales hoc modo:

$1+Az+Bz^2+Cz^3+\ldots+Kz^{n-3}+Lz^{n-2}+Mz^{n-1}+Nz^n+$ &c., ac quilibet coëfficiens N ex tot procedentibus, quot ſunt litteræ $\alpha, \beta, \gamma, \delta$, &c. ita determinabitur ut ſit: $N = \frac{m+n}{n}\alpha M + \frac{2m+n}{n}\beta L + \frac{3m+n}{n}\gamma K + \frac{4m+n}{n}\delta I +$ &c. quæ lex continuationis etſi non eſt conſtans, ſed ab exponente poteſtatis z pendet, tamen eidem Seriei alia convenit lex progreſſionis conſtans, quam denominator evolutus præbet,

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Lib. I. naturæ Serierum recurrentium conſentaneam. Illa vero lex non conſtans tantum locum habet ſi numerator fractionis fuerit unitas ſeu quantitas conſtans; ſi enim quoque aliquot poteſtates ipſius z contineret, tum illa lex multo magis fieret complicata, id quod poſt tradita calculi differentialis principia facilius patebit.

69. Quoniam hactenus poſuimus primum denominatoris terminum conſtantem non eſſe $= 0$, ejuſque loco unitatem collocavimus; nunc videamus cujuſmodi Series oriantur, ſi in denominatore terminus conſtans evaneſcat. His caſibus ergo Functio fracta hujuſmodi formam habebit $\frac{a + bz + czz + \&c.}{z(1 - \alpha z - \beta zz - \&c.)}$, convertatur ergo, neglecto denominatoris Factore z, reliqua fractio $\frac{a + bz + czz + \&c.}{1 - \alpha z - \beta z^2 - \&c.}$ in Seriem recurrentem $A + Bz + Cz^2 + Dz^3 + \&c.$ atque, facta diviſione per z, manifeſtum eſt fore $\frac{a + bz + czz + \&c.}{z(1 - \alpha z - \beta z^2 - \gamma z^3 - \&c.)} = \frac{A}{z} + B + Cz + Dz^2 + Ez^3 + \&c.$ Simili modo erit $\frac{a + bz + cz^2 + \&c.}{z^2(1 - \alpha z - \beta z^2 - \&c.)} = \frac{A}{zz} + \frac{B}{z} + C + Dz + Ez^2 + \&c.$, atque generatim erit $\frac{a + bz + cz^2 + \&c.}{z^m(1 - \alpha z - \beta z^2 - \gamma z^3 - \&c.)} = \frac{A}{z^m} + \frac{B}{z^{m-1}} + \frac{C}{z^{m-2}} + \frac{D}{z^{m-3}} + \&c.$ quicunque numerus fuerit exponens m.

70. Quoniam per ſubſtitutionem loco z alia variabilis x in Functionem fractam introduci, hocque pacto Functio fracta quævis in innumerabiles formas diverſas transmutari poteſt; hoc modo eadem Functio fracta infinitis modis per Series recurrentes explicari poterit. Sit ſcilicet propoſita hæc fractio $y = \frac{1 + z}{1 - z - zz}$ & per Seriem recurrentem $y = 1 + 2z + 3z^2 + 5z^3 + 8z^4 + \&c.$: ponatur $z = \frac{1}{x}$ erit y

$= \frac{xx + x}{xx - x - 1} = \frac{-x(1+x)}{1 + x - xx}$. Jam $\frac{1+x}{1+x-xx} = 1 + 0x + xx - x^3 + 2x^4 - 3x^5 + 5x^6 -$ &c.; unde erit $y = -x + 0x^2 - x^3 + x^4 - 2x^5 + 3x^6 - 5x^7 +$ &c. Vel ponatur $z = \frac{1-x}{1+x}$, erit $y = \frac{-2-2x}{1-4x-xx}$; unde fit $y = -2 - 10x - 42xx - 178x^3 - 754x^4 -$ &c. cujuſmodi Series recurrentes pro y innumerabiles inveniri poſſunt.

71. Functiones irrationales ex hoc theoremate univerſali in Series infinitas transformari ſolent, quod ſit $(P+Q)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} P^{\frac{m-n}{n}} Q + \frac{m(m-n)}{n.\ 2n} P^{\frac{m-2n}{n}} Q^2 + \frac{m(m-n)(m-2n)}{n.\ 2n.\ 3n} P^{\frac{m-3n}{n}} Q^3 +$ &c.: hi enim termini, niſi fuerit $\frac{m}{n}$ numerus integer affirmativus, in infinitum excurrunt. Sic erit pro m & n numeros definitos ſcribendo.

$(P+Q)^{\frac{1}{2}} = P^{\frac{1}{2}} + \frac{1}{2} P^{-\frac{1}{2}} Q - \frac{1.\ 1}{2.\ 4} P^{-\frac{3}{2}} Q^2 + \frac{1.\ 1.\ 3}{2.\ 4.\ 6} P^{-\frac{5}{2}} Q^3 -$ &c.

$(P+Q)^{-\frac{1}{2}} = P^{-\frac{1}{2}} - \frac{1}{2} P^{-\frac{3}{2}} Q + \frac{1.\ 3}{2.\ 4} P^{-\frac{5}{2}} Q^2 - \frac{1.\ 3.\ 5}{2.\ 4.\ 6} P^{-\frac{7}{2}} Q^3 +$ &c.

$(P+Q)^{\frac{1}{3}} = P^{\frac{1}{3}} + \frac{1}{3} P^{-\frac{2}{3}} Q - \frac{1.\ 2}{3.\ 6} P^{-\frac{5}{3}} Q^2 + \frac{1.\ 2.\ 5}{3.\ 6.\ 9} P^{-\frac{8}{3}} Q^3 -$ &c.

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$(P+Q)^{-\frac{1}{3}} = P^{-\frac{1}{3}} - \frac{1}{3} P^{-\frac{4}{3}} Q + \frac{1.\ 4}{3.\ 6} P^{-\frac{7}{3}} Q^2 - \frac{1.\ 4.\ 7}{3.\ 6.\ 9} P^{-\frac{10}{3}} Q^3 + \&c.$

$(P+Q)^{\frac{2}{3}} = P^{\frac{2}{3}} + \frac{2}{3} P^{-\frac{1}{3}} Q - \frac{2.\ 1}{3.\ 6} P^{-\frac{4}{3}} Q^2 + \frac{2.\ 1.\ 4}{3.\ 6.\ 9} P^{-\frac{7}{3}} Q^3 - \&c.$

&c.

72. Hujuſmodi ergo Serierum termini ita progrediuntur ut quilibet ex antecedente formari poſſit : ſit enim Seriei, quæ ex $(P+Q)^{\frac{m}{n}}$ naſcitur, terminus quilibet $= MP^{\frac{m-kn}{n}} Q^k$ erit ſequens $= \frac{m-kn}{(k+1)n} MP^{\frac{m-(k+1)n}{n}} Q^{k+1}$. Notandum autem eſt in quovis termino ſequente exponentem ipſius P unitate decreſcere, contra vero exponentem ipſius Q unitate creſcere. Quo autem hæc facilius ad quemvis caſum accommodentur, forma generalis $(P+Q)^{\frac{m}{n}}$ ita exponi poteſt $P^{\frac{m}{n}}(1+\frac{Q}{P})^{\frac{m}{n}}$: evoluta enim formula $(1+\frac{Q}{P})^{\frac{m}{n}}$ Serieque reſultante per $P^{\frac{m}{n}}$ multiplicata, prodibit ipſa Series ante data. Tum vero ſi m non ſolum numeros integros denotet, ſed etiam fractos, pro n tuto unitas collocari poterit. Quibus factis, ſi pro $\frac{Q}{P}$, quæ eſt Functio ipſius z, ponatur Z, habebitur $(1+Z)^m = 1 + \frac{m}{1} Z + \frac{m(m-1)}{1.\ 2} Z^2 + \frac{m(m-1)(m-2)}{1.\ 2.\ 3} Z^3 + \&c.$ Ad ſequentes progreſſionum leges autem obſervandas conveniet hanc formulæ generalis in Seriem converſionem notaſſe

taſſe $(1+Z)^{m-1} = 1 + \frac{(m-1)}{1}Z + \frac{(m-1)(m-2)}{1.\ 2}Z^2 + \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3}Z^3 + \&c.$

73. Sit igitur primum $Z = \alpha z$, eritque $(1+\alpha z)^{m-1} = 1 + \frac{m-1}{1}\alpha z + \frac{(m-1)(m-2)}{1.\ 2}\alpha^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3}\alpha^3 z^3 + \&c.$ Scribatur pro hac Serie iſta forma generalis

$1 + Az + Bz^2 + Cz^3 + \ldots\ldots + Mz^{n-1} + Nz^n + \&c.$

atque quilibet coëfficiens N ex præcedente M ita determinabitur ut ſit $N = \frac{m-n}{n}\alpha M.$ Sic, poſito $n = 1$, cum ſit $M = 1$, erit $N = A = \frac{m-1}{1}\alpha$; tum facto $n = 2$, ob $M = A = \frac{m-1}{1}\alpha$, erit $N = B = \frac{m-2}{2}\alpha M = \frac{(m-1)(m-2)}{1.\ 2}\alpha^2$; ſimilique modo porro $C = \frac{m-3}{3}\alpha B = \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3}\alpha^3$, uti Series ante inventa declarat.

74. Sit $Z = \alpha z + \beta zz$, eritque $(1 + \alpha z + \beta zz)^{m-1} = 1 + \frac{(m-1)}{1}(\alpha z + \beta zz) + \frac{(m-1)(m-2)}{1.\ 2}(\alpha z + \beta zz)^2 + \&c.$ Quod ſi ergo termini ſecundum poteſtates ipſius z diſponantur, erit $(1 + \alpha z + \beta zz)^{m-1} =$

$$1 + \frac{(m-1)}{1}\alpha z + \frac{(m-1)(m-2)}{1.\ 2}\alpha^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3}\alpha^3 z^3 + \&c.$$
$$+ \frac{(m-1)}{1}\beta z^2 + \frac{(m-1)(m-2)}{1.\ 2}2\alpha\beta z^3 + \&c.$$

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LIB. I. Scribatur pro hac Serie iſta forma generalis:

$$1 + Az + Bz^2 + Cz^3 + \ldots + Lz^{n-2} + Mz^{n-1} + Nz^n + \&c.$$

atque quilibet coëfficiens ex duobus antecedentibus ita definietur ut ſit $N = \frac{m-n}{n}\,\alpha M + \frac{2m-n}{n}\,\beta L$, unde omnes termini ex primo, qui eſt 1, definiri poterunt. Erit nempe

$$\begin{aligned}
A &= \frac{m-1}{1}\,\alpha\,;\\
B &= \frac{(m-2)}{2}\,\alpha A + \frac{(2m-2)}{2}\,\beta\\
C &= \frac{(m-3)}{3}\,\alpha B + \frac{(2m-3)}{3}\,\beta A\\
D &= \frac{(m-4)}{4}\,\alpha C + \frac{(2m-4)}{4}\,\beta B\\
&\quad \&c.
\end{aligned}$$

75. Si fuerit $Z = \alpha z + \beta z^2 + \gamma z^3$, erit $(1 + \alpha z + \beta z^2 + \gamma z^3)^{m-1} = 1 + \frac{(m-1)}{1}(\alpha z + \beta z^2 + \gamma z^3) + \frac{(m-1)(m-2)}{1.\ 2}(\alpha z + \beta z^2 + \gamma z^3)^2 + \&c.$, quæ expreſſio, ſi omnes termini ſecundum poteſtates ipſius z ordinentur, abibit in hanc Seriem:

$$\begin{aligned}
1 + \frac{(m-1)}{1}\alpha z + \frac{(m-1)(m-2)}{1.\ 2}\alpha^2 z^2 &+ \frac{(m-1)(m-2)(m-3)}{1.\ 2.\ 3}\alpha^3 z^3\\
+ \frac{(m-1)}{1}\beta z^2 &+ \frac{(m-1)(m-2)}{1.\ 2}2\alpha\beta z^3 + \&c.\\
&+ \frac{(m-1)}{1}\gamma z^3
\end{aligned}$$

cujus lex progreſſionis ut melius pateſcat, ponatur ejus loco $1 + Az + Bz^2 + Cz^3 + \ldots + Kz^{n-3} + Lz^{n-2} + Mz^{n-1} + Nz^n$, cujus Seriei quilibet coëfficiens ex tribus antecedentibus ita determinatur ut ſit $N = \frac{(m-n)}{n}\alpha M + \frac{(2m-n)}{n}\beta L + \frac{(3m-n)}{n}\gamma K$.

Cum igitur primus terminus ſit $= 1$, & antecedentes nulli, erit

$$A = \frac{m-1}{1}\alpha$$

$$B = \frac{(m-2)}{2}\alpha A + \frac{(2m-2)}{2}\beta$$

$$C = \frac{(m-3)}{3}\alpha B + \frac{(2m-3)}{3}\beta A + \frac{(3m-3)}{3}\gamma$$

$$D = \frac{(m-4)}{4}\alpha C + \frac{(2m-4)}{4}\beta B + \frac{(3m-4)}{4}\gamma A$$

$$E = \frac{(m-5)}{5}\alpha D + \frac{(2m-5)}{5}\beta C + \frac{(3m-5)}{5}\gamma B$$

&c.

76. Generaliter ergo ſi ponatur $(1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \&c.)^{m-1} = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \&c.$, hujus Seriei ſinguli termini ita ex præcedentibus definientur, ut ſit

$$A = \frac{m-1}{1}\alpha$$

$$B = \frac{(m-2)}{2}\alpha A + \frac{(2m-2)}{2}\beta$$

$$C = \frac{(m-3)}{3}\alpha B + \frac{(2m-3)}{3}\beta A + \frac{(3m-3)}{3}\gamma$$

$$D = \frac{(m-4)}{4}\alpha C + \frac{(2m-4)}{4}\beta B + \frac{(3m-4)}{4}\gamma A + \frac{(4m-4)}{4}\delta$$

$$E = \frac{(m-5)}{5}\alpha D + \frac{(2m-5)}{5}\beta C + \frac{(3m-5)}{5}\gamma B + \frac{(4m-5)}{5}\delta A + \frac{(5m-5)}{5}\epsilon$$

&c.

quilibet ſcilicet terminus per tot præcedentes determinatur, quot habentur litteræ α, β, γ, δ, &c. in Functione ipſius z cujus poteſtas in Seriem convertitur. Ceterum ratio hujus legis convenit cum ea, quam ſupra §. 68. ubi ſimilem formam $(1 - \alpha z - \beta z^2 - \gamma z^3 - \&c.)^{-m-1}$ in Seriem infini-

 tam reſolvimus; ſi enim loco m ſcribatur $-m$ atque litteræ α, β, γ, δ, &c. negative accipiantur, Series inventæ prorſus congruent. Interim hoc loco non licet rationem hujus progreſſionis legis a priori demonſtrare, id quod per principia calculi differentialis demum commode fieri poterit; interea ergo ſufficiet veritatem per applicationem ad omnis generis exempla comprobaſſe.

CAPUT V.

De Functionibus duarum pluriumve variabilium.

77. Quanquam plures hactenus quantitates variabiles ſumus contemplati, tamen eæ ita erant comparatæ, ut omnes unius eſſent Functiones, unaque determinata reliquæ ſimul determinarentur. Nunc autem ejuſmodi conſiderabimus quantitates variabiles, quæ a ſe invicem non pendeant, ita ut quamvis uni determinatus valor tribuatur, reliquæ tamen nihilominus maneant indeterminatæ ac variabiles. Ejuſmodi ergo quantitates variabiles, cujuſmodi ſint x, y, z, ratione ſignificationis convenient, cum quælibet omnes valores determinatos in ſe complectatur; at, ſi inter ſe comparentur maxime erunt diverſæ, cum, licet pro una z valor quicunque determinatus ſubſtituatur, reliquæ tamen x & y æque late pateant, atque ante. Diſcrimen ergo inter quantitates variabiles a ſe pendentes, & non pendentes in hoc verſatur, ut priori caſu, ſi una determinetur, ſimul reliquæ determinentur; poſteriori vero determinatio unius ſignificationes reliquarum minime reſtringat.

78. *Functio ergo duarum pluriumve quantitatum variabilium*, x, y, z, *eſt expreſſio quomodocunque ex his quantitatibus compoſita.*

Ita erit $x^3 + xyz + az^2$ Functio quantitatum variabilium trium x, y, z. Hæc ergo Functio, ſi una determinetur va-

riabilis, puta z, hoc eſt ejus loco conſtans numerus ſubſtituatur, manebit adhuc quantitas variabilis, ſcilicet Functio ipſarum x & y. Atque ſi, præter z, quoque y determinetur, tum erit adhuc Functio ipſius x. Hujuſmodi ergo plurium variabilium Functio non ante valorem determinatum obtinebit, quam ſingulæ quantitates variabiles fuerint determinatæ. Cum igitur una quantitas variabilis infinitis modis determinari poſſit, Functio duarum variabilium, quia pro quavis determinatione unius infinitas determinationes ſuſcipere poteſt, omnino infinities infinitas determinationes admittet. Atque in Functione trium variabilium numerus determinationum erit adhuc infinities major; ſicque porro creſcet pro pluribus variabilibus.

79. *Hujuſmodi Functiones plurium variabilium perinde atque Functiones unius variabilis, commodiſſime dividuntur in algebraicas ac tranſcendentes.*

Quarum illæ ſunt, in quibus ratio compoſitionis in ſolis Algebræ operationibus eſt poſita; hæ vero, in quarum formationem quoque operationes tranſcendentes ingrediuntur. In his denuo ſpecies notari poſſent, prout operationes tranſcendentes vel omnes quantitates variabiles implicant, vel aliquot, vel tantum unicam. Sic iſta expreſſio $zz + y \log. z$, quia Logarithmus ipſius z ineſt, erit quidem Functio tranſcendens ipſarum y & z, verum ideo minus tranſcendens eſt putanda, quod ſi variabilis z determinetur, ſuperſit Functio algebraica ipſius y. Interim tamen non expedit hujuſmodi ſubdiviſionibus tractationem amplificari.

80. *Functiones deinde algebraicæ ſubdividuntur in rationales & irrationales; rationales autem porro in integras ac fractas.*

Ratio harum denominationum ex Capite primo jam abunde intelligitur. Functio ſcilicet rationalis omnino eſt libera ab omni irrationalitate quantitates variabiles, quarum Functio dicitur, afficiente; hæcque erit integra ſi nullis fractionibus inquinetur, contra vero fracta. Sic functionis integræ duarum variabilium y & z hæc erit forma generalis: $\alpha + \beta y + \gamma z + \delta y^2 + \epsilon yz + \zeta z^2 + \eta y^3 + \theta y^2 z + \iota y z^2 + \kappa z^3 +$ &c. Quod

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 ſi ergo P & Q denotent hujuſmodi Functiones integras, ſive duarum ſive plurium variabilium, erit $\frac{P}{Q}$ forma generalis Functionum fractarum. Functio denique irrationalis eſt vel explicita, vel implicita; illa per ſigna radicalia jam penitus eſt evoluta, hæc autem per æquationem irreſolubilem exhibetur: ſic V erit Functio implicita irrationalis ipſarum y & z, ſi fuerit $V^3 = (ayz + z^3)V^2 + (y^4 + z^4)V + y^5 + 2ayz^3 + z^5$.

81. *Multiformitas deinde in his Functionibus æque notari debet, atque in iis, quæ ex unica variabili conſtant.*

Sic Functiones rationales erunt uniformes, quia ſingulis quantitatibus variabilibus determinatis, unicum valorem determinatum exhibent. Denotent P, Q, R, S, &c. Functiones rationales ſeu uniformes variabilium x, y, z, eritque V Functio biformis earundem variabilium, ſi fuerit $V^2 - PV + Q = 0$; quicunque enim valores determinati quantitatibus x, y, & z tribuantur, Functio V non unum ſed duplicem perpetuo habebit valorem determinatum. Simili modo erit V Functio triformis ſi fuerit $V^3 - PV^2 + QV - R = 0$: atque Functio quadriformis ſi fuerit $V^4 - PV^3 + QV^2 - RV + S = 0$: hocque modo ratio Functionum multiformium ulteriorum erit comparata.

82. Quemadmodum ſi Functio unius variabilis z nihilo æqualis ponitur, quantitas variabilis z valorem conſequitur determinatum vel ſimplicem vel multiplicem; ita ſi Functio duarum variabilium y & z nihilo æqualis ponitur, tum altera variabilis per alteram definitur, ejuſque ideo Functio evadit, cum ante a ſe mutuo non penderent. Simili modo ſi Functio trium variabilium x, y, z, nihilo æqualis ſtatuatur, tum una variabilis per duas reliquas definitur, earumque Functio exiſtit. Idem evenit ſi functio non nihilo ſed quantitati conſtanti vel etiam alii Functioni æqualis ponatur; ex omni enim æquatione, quotcunque variabiles involvat, ſemper una variabilis per reliquas definitur earumque fit Functio; duæ autem æqua-

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tiones diverſæ inter eaſdem variabiles ortæ binas per reliquas definient, atque ita porro. CAP. V.

83. *Functionum autem duarum pluriumve variabilium diviſio maxime notatu digna eſt in homogeneas & heterogeneas.*

Functio homogenea eſt per quam ubique idem regnat variabilium numerus dimenſionum: Functio autem heterogenea eſt, in qua diverſi occurrunt dimenſionum numeri. Cenſetur vero unaquæque variabilis unam dimenſionem conſtituere; quadratum uniuſcujuſque atque productum ex duabus, duas; productum ex tribus variabilibus, ſive iiſdem ſive diverſis, tres & ita porro; quantitates autem conſtantes ad dimenſionum numerationem non admittuntur. Ita in his formulis αy; βz, unica dimenſio ineſſe dicitur; in his vero αy^2; βyz; γz^2 duæ inſunt dimenſiones: in his αy^3; $\beta y^2 z$; γyz^2; δz^3, tres; in his vero αy^4; $\beta y^3 z$; $\gamma y^2 z^2$; δyz^3; ϵz^4; quatuor, ſicque porro.

84. Applicemus primum hanc diſtinctionem ad Functiones integras, atque duas tantum variabiles ineſſe ponamus, quoniam plurium par eſt ratio.

Functio igitur integra erit homogenea in cujus ſingulis terminis idem exiſtit dimenſionum numerus.

Subdividentur ergo hujuſmodi Functiones commodiſſime ſecundum numerum dimenſionum, quem variabiles in ipſis ubique conſtituunt. Sic erit $\alpha y + \beta z$ forma generalis Functionum integrarum unius dimenſionis: hæc vero expreſſio $\alpha y^2 + \beta yz + \gamma z^2$ erit forma generalis Functionum duarum dimenſionum, tum forma generalis Functionum trium dimenſionum erit: $\alpha y^3 + \beta y^2 z + \gamma yz^2 + \delta z^3$; quatuor dimenſionum vero hæc: $\alpha y^4 + \beta y^3 z + \gamma y^2 z^2 + \delta yz^3 + \epsilon z^4$; & ita porro. Ad analogiam igitur erit quantitas conſtans ſola α Functio nullius dimenſionis.

85. *Functio porro fracta erit homogenea, ſi ejus Numerator ac Denominator fuerint Functiones homogeneæ.*

Sic hæc Fractio $\frac{ayy + bzz}{\alpha y + \beta z}$ erit Functio homogenea ipſa-

 rum y & z; numerus dimenſionum autem habebitur, ſi a numero dimenſionum Numeratoris ſubtrahatur numerus dimenſionum Denominatoris: atque ob hanc rationem Fractio allata erit Functio unius dimenſionis. Hæc vero Fractio $\frac{y^5+z^5}{yy+zz}$ erit Functio trium dimenſionum. Quando ergo in Numeratore ac Denominatore idem dimenſionum numerus ineſt, tum Fractio erit Functio nullius dimenſionis, uti evenit in hac Fractione $\frac{y^3+z^3}{yyz}$, vel etiam in his $\frac{y}{z}$; $\frac{zz}{yy}$; $\frac{Cy^3}{z^3}$. Quod ſi igitur in Denominatore plures ſint dimenſiones quam in Numeratore, numerus dimenſionum Fractionis erit negativus: ſic $\frac{y}{zz}$ erit Functio — 1 dimenſionis: $\frac{y+z}{y^4+z^4}$ erit Functio — 3 dimenſionum: $\frac{1}{y^5+ayz^4}$ erit Functio — 5 dimenſionum, quia in Numeratore nulla ineſt dimenſio. Ceterum ſponte intelligitur plures Functiones homogeneas, in quibus ſingulis idem regnat dimenſionum numerus, ſive additas ſive ſubtractas præbere Functionem quoque homogeneam ejuſdem dimenſionum numeri. Sic hæc expreſſio $\alpha y + \frac{\beta zz}{y} + \frac{\gamma y^4 - \delta z^4}{yyz+yzz}$ erit Functio unius dimenſionis: hæc autem $\alpha + \frac{\beta y}{z} + \frac{\gamma zz}{yy} + \frac{yy+zz}{yy-zz}$ erit Functio nullius dimenſionis.

86. Natura Functionum homogenearum quoque ad expreſſiones irrationales extenditur. Si enim fuerit P Functio quæcunque homogenea, puta n dimenſionum, tum $\sqrt{P}$ erit Functio $\frac{1}{2}n$ dimenſionum; $\sqrt[3]{P}$ erit Functio $\frac{1}{3}n$ dimenſionum, & generatim $P^{\frac{\mu}{\nu}}$ erit Functio $\frac{\mu}{\nu}n$ dimenſionum. Sic $\sqrt{(yy+zz)}$ erit Functio unius dimenſionis; $\sqrt[3]{(y^9+z^9)}$ erit Functio trium dimenſionum: $(yz+zz)^{\frac{3}{4}}$ erit Functio $\frac{3}{2}$ dimenſionum:

atque

atque $\frac{yy+zz}{\sqrt{(y^4+z^4)}}$ erit Functio nullius dimenſionis. His ergo CAP. V. cum præcedentibus conjunctis intelligitur hæc expreſſio $\frac{1}{y} + \frac{y\sqrt{(yy+zz)}}{z^3} - \frac{y}{\sqrt[3]{(y^6-z^6)}} + \frac{y\sqrt{z}}{zz\sqrt{y}+\sqrt{(y^5+z^5)}}$ eſſe Functio homogenea -1 dimenſionis.

87. Utrum Functio irrationalis implicita ſit homogenea necne, ex his facile colligi poteſt. Sit V hujuſmodi Functio implicita ac $V^3 + PV^2 + QV + R = 0$, exiſtentibus P, Q & R Functionibus ipſarum y & z. Primum igitur patet V Functionem homogeneam eſſe non poſſe, niſi P, Q, & R ſint Functiones homogeneæ. Præterea vero ſi ponamus V eſſe Functionem n dimenſionum, erit V^2 Functio $2n$, & V^3 Functio $3n$ dimenſionum; cum igitur ubique idem debeat eſſe numerus dimenſionum, oportet, ut P ſit Functio n dimenſionum, Q Functio $2n$ dimenſionum, & R Functio $3n$ dimenſionum. Sint ergo viciſſim litteræ P, Q, R Functiones homogeneæ reſpective n, $2n$, $3n$ dimenſionum, hinc concludetur fore V Functionem n dimenſionum. Ita ſi fuerit $V^5 + (y^4+z^4)V^3 + ay^8V - z^{10} = 0$ erit V Functio homogenea duarum dimenſionum, ipſarum y & z.

88. *Si fuerit* V *Functio homogenea* n *dimenſionum ipſarum* y *&* z, *in eaque ponatur ubique* $y = uz$, *Functio* V *abibit in productum ex poteſtate* z^n *in Functionem quandam variabilis* u.

Per hanc enim ſubſtitutionem $y = uz$, in ſingulos terminos tantæ inducentur poteſtates ipſius z, quantæ ante inerant ipſius y. Cum igitur in ſingulis terminis dimenſiones ipſarum y & z conjunctim æquaſſent numerum n, nunc ſola variabilis z ubique habebit n dimenſiones, ideoque ubique inerit ejus poteſtas z^n. Per hanc ergo poteſtatem Functio V fiet diviſibilis & quotus erit Functio variabilem tantum u involvens. Hoc primum patebit in Functionibus integris; ſi enim ſit $V = \alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3$, poſito $y = uz$, fiet $V = z^3$

LIB. I. $(\alpha u^3 + \beta u^2 + \gamma u + \delta)$. Deinde vero idem manifeſtum eſt in fractis: ſit enim $V = \frac{\alpha y + \beta z}{yy + zz}$, nempe Functio -1 dimenſionis, facto $y = uz$ fiet $V = z^{-1} \left(\frac{\alpha u + \beta}{uu + 1}\right)$. Neque etiam Functiones irrationales hinc excipiuntur, ſi enim ſit $V = \frac{y + \sqrt{(yy + zz)}}{z\sqrt{(y^3 + z^3)}}$, quæ eſt Functio $-\frac{3}{2}$ dimenſionum; poſito $y = uz$, prodibit $V = z^{-\frac{3}{2}} \left(\frac{u + \sqrt{(uu + 1)}}{\sqrt{(u^3 + 1)}}\right)$. Hoc itaque modo Functiones homogeneæ duarum tantum variabilium reducentur ad Functiones unius variabilis; neque enim poteſtas ipſius z, quia eſt Factor, Functionem illam ipſius u inquinat.

89. *Functio ergo homogenea* V *duarum variabilium* y & z *nullius dimenſionis, poſito* y = u z, *tranſmutabitur in Functionem unicæ variabilis* u *puram.*

Cum enim numerus dimenſionum ſit nullus, Poteſtas ipſius z, quæ Functionem ipſius u multiplicabit, erit $z^0 = 1$; hocque caſu variabilis z prorſus ex computo egredietur. Ita ſi fuerit $V = \frac{y + z}{y - z}$, facto $y = uz$, orietur $V = \frac{u + 1}{u - 1}$: atque in irrationalibus ſi ſit $V = \frac{y - \sqrt{(yy - zz)}}{z}$ poſito $y = uz$ erit $V = u - \sqrt{(uu - 1)}$.

90. *Functio integra homogenea duarum variabilium* y & z, *reſolvi poterit in tot Factores ſimplices formæ* $\alpha y + \beta z$, *quot habuerit dimenſiones.*

Cum enim Functio ſit homogenea, poſito $y = uz$, tranſibit in productum ex z^n in Functionem quandam ipſius u integram, quæ Functio propterea in Factores ſimplices formæ $\alpha u + \beta$ reſolvi poterit. Multiplicentur ſinguli Factores hi per z, eritque uniuſcujuſque forma $\alpha uz + \beta z = \alpha y + \beta z$ ob $uz = y$. Propter multiplicatorem autem z^n, tot hujuſmodi Factores naſcentur quot exponens n contineat unitates; Factores autem

hi ſimplices erunt vel reales vel imaginarii ; hoc eſt, coëfficientes α & β erunt vel reales vel imaginarii.

Ex hoc itaque ſequitur Functionem duarum dimenſionum $ayy + byz + czz$ duos habere Factores ſimplices formæ $\alpha y + \beta z$; Functio autem $ay^3 + by^2z + cyz^2 + dz^3$ habebit tres Factores ſimplices formæ $\alpha y + \beta z$; ſicque porro Functionum homogenearum integrarum, quæ plures habent dimenſiones, natura erit comparata.

91. Quemadmodum ergo hæc expreſſio $\alpha y + \beta z$ continet formam generalem Functionum integrarum unius dimenſionis, ita $(\alpha y + \beta z)(\gamma y + \delta z)$ erit forma generalis Functionum integrarum duarum dimenſionum : atque in hac forma $(\alpha y + \beta z)(\gamma y + \delta z)(\epsilon y + \xi z)$ continebuntur omnes Functiones integræ trium dimenſionum, ſicque omnes Functiones integræ homogeneæ per producta ex tot hujuſmodi Factoribus $\alpha y + \beta z$ exhiberi poterunt, quot Functiones illæ contineant dimenſiones. Iſti autem Factores eodem modo per reſolutionem æquationum reperiuntur, quo ſupra Factores ſimplices Functionum integrarum unius variabilis invenire docuimus. Ceterum hæc proprietas Functionum homogenearum duarum variabilium non extenditur ad Functiones homogeneas trium pluriumve variabilium : forma enim generalis hujuſmodi Functionum duarum tantum dimenſionum, quæ eſt $ayy + byz + cyx + dxy + exx + fzz$ generaliter non reduci poteſt ad hujuſmodi productum $(\alpha y + \beta z + \gamma x)(\delta y + \epsilon z + \xi x)$; multoque minus Functiones plurium dimenſionum ad hujuſmodi producta revocari poſſunt.

92. Ex his, quæ de Functionibus homogeneis ſunt dicta, ſimul intelligitur, quid ſit Functio heterogenea : in cujus ſcilicet terminis non ubique idem dimenſionum numerus deprehenditur. Poſſunt autem Functiones heterogeneæ ſubdividi pro multiplicitate dimenſionum, quæ in ipſis occurrunt. Sic Functio bifida erit, in qua duplex dimenſionum numerus occurrit, eritque adeo aggregatum duarum Functionum homo-

I 2

 genearum, quarum numeri dimenſionum differunt; ita $y^5 + 2y^3z^2 + yy + zz$ erit Functio bifida, quia partim quinque, partim duas continet dimenſiones. Functio autem trifida eſt, in qua tres diverſi dimenſionum numeri inſunt, ſeu quæ in tres Functiones homogeneas diſtribui poteſt, uti $y^6 + y^4z^2 + z^4 + y - z$.

Præterea autem dantur Functiones heterogeneæ fractæ vel irrationales tantopere permixtæ, quæ in Functiones homogeneas reſolvi non poſſunt, cujuſmodi ſunt $\frac{y^3 + ayz}{by + zz}$, $\frac{a + \surd(yy + zz)}{yy - bz}$.

93. Interdum Functio heterogenea ope ſubſtitutionis idoneæ, vel loco unius vel utriuſque variabilis factæ, ad homogeneam reduci poteſt; quod quibus caſibus fieri queat, non tam facile indicare licet. Sufficiet ergo exempla quædam attuliſſe, quibus ejuſmodi reductio locum habet. Si ſcilicet hæc propoſita ſit Functio $y^5 + zzy + y^3z + \frac{z^3}{y}$; poſt levem attentionem apparebit, eam ad homogeneitatem perduci, poſito $z = xx$: prodibit enim $y^5 + x^4y + y^3xx + \frac{x^6}{y}$, Functio homogenea 5 dimenſionum ipſarum x & y. Deinde hæc Functio $y + y^2x + y^3xx + y^5x^4 + \frac{a}{x}$ ad homogeneitatem reducitur ponendo $x = \frac{1}{z}$, prodit enim Functio unius dimenſionis $y + \frac{yy}{z} + \frac{y^3}{zz} + \frac{y^5}{z^4} + az$. Multo difficiliores autem ſunt caſus, quibus non per tam ſimplicem ſubſtitutionem ad homogeneitatem pervenire licet.

94. Tandem inprimis notari meretur Functionum integrarum ſecundum ordines diviſio ſatis uſitata, ſecundum quam ordo definitur ex maximo dimenſionum numero qui in Functione ineſt. Sic $xx + yy + zz + ay - aa$ eſt Functio ſecundi ordinis, quia duæ dimenſiones occurrunt. Et $y^4 + yz^3 - ay^2z +$

$abyz - aayy + b^4$ pertinet ad Functiones quarti ordinis. Ad hanc diviſionem potiſſimum in doctrina de lineis curvis reſpici ſolet; unde adhuc una Functionum integrarum diviſio commemoranda venit.

95. Supereſt ſcilicet diviſio Functionum integrarum in complexas atque incomplexas. Functio autem complexa eſt, quæ in Factores rationales reſolvi poteſt, ſeu quæ eſt productum ex duabus Pluribuſve functionibus rationalibus; cujuſmodi eſt $y^4 - z^4 + 2az^3 - 2byzz - aazz + 2abzy - bbyy$, quæ eſt productum ex his duabus Functionibus $(yy + zz - az + by)$ $(yy - zz + az - by)$. Ita vidimus omnem Functionem integram homogeneam, quæ tantum duas variabiles complectatur, eſſe Functionem complexam, quoniam tot Factores ſimplices formæ $\alpha y + \beta z$ habet, quot continet dimenſiones. Functio igitur integra erit incomplexa, ſi in Factores rationales reſolvi omnino nequeat; uti $yy + zz - aa$, cujus nullos dari Factores rationales facile intelligitur. Ex inquiſitione Diviſorum patebit, utrum Functio propoſita ſit complexa an incomplexa.

CAPUT VI.

De Quantitatibus exponentialibus ac Logarithmis.

96. Quanquam notio Functionum tranſcendentium in calculo integrali demum perpendetur, tamen antequam eo perveniamus, quaſdam ſpecies magis obvias, atque ad plures inveſtigationes aditum aperientes, evolvere conveniet. Primum ergo conſiderandæ ſunt quantitates exponentiales, ſeu Poteſtates, quarum Exponens ipſe eſt quantitas variabilis. Perſpicuum enim eſt hujuſmodi quantitates ad Functiones algebraicas referri non poſſe, cum in his Exponentes non niſi conſtantes locum habeant. Multiplices autem ſunt quantitates exponen-

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tiales, prout vel ſolus Exponens eſt quantitas variabilis, vel præterea etiam ipſa quantitas elevata; prioris generis eſt a^z, hujus vero y^z; quin etiam ipſe Exponens poteſt eſſe quantitas exponentialis uti in his formis a^{a^z}; a^{y^z}; y^{a^z}; x^{y^z}. Hujuſmodi autem quantitatum non plura conſtituemus genera, cum earum natura ſatis clarè intelligi queat, ſi primam tantum ſpeciem a^z evolverimus.

97. Sit igitur propoſita hujuſmodi quantitas exponentialis a^z, quæ eſt Poteſtas quantitatis conſtantis a, Exponentem habens variabilem z. Cum igitur iſte Exponens z omnes numeros determinatos in ſe complectatur, primum patet ſi loco z omnes numeri integri affirmativi ſucceſſive ſubſtituantur, loco a^z hos prodituros eſſe valores determinatos a^1; a^2; a^3; a^4; a^5; a^6; &c. Sin autem pro z ponantur ſucceſſive numeri negativi -1, -2, -3, &c. prodibunt $\frac{1}{a}$; $\frac{1}{a^2}$; $\frac{1}{a^3}$; $\frac{1}{a^4}$; &c. ac, ſi fuerit $z = 0$, habebitur ſemper $a^0 = 1$. Quod ſi loco z numeri fracti ponantur, ut $\frac{1}{2}$; $\frac{1}{3}$; $\frac{2}{3}$; $\frac{1}{4}$; $\frac{3}{4}$; &c. orientur iſti valores $\sqrt{a}$; $\sqrt[3]{a}$; $\sqrt[3]{aa}$; $\sqrt[4]{a}$; $\sqrt[4]{a^3}$; &c., qui in ſe ſpectati geminos pluresve induunt valores, cum radicum extractio ſemper valores multiformes producat. Interim tamen hoc loco valores tantum primarii, reales ſcilicet atque affirmativi admitti ſolent; quia quantitas a^z tanquam Functio uniformis ipſius z ſpectatur. Sic $a^{\frac{5}{2}}$ medium quemdam tenebit locum inter a^2 & a^3, eritque ideo quantitas ejuſdem generis; & quamvis valor $a^{\frac{5}{2}}$ ſit æque $= - aa\sqrt{a}$, ac $= + aa\sqrt{a}$, tamen poſterior tantum in cenſum venit. Eodem modo res ſe habet, ſi Exponens z valores irrationales accipiat, quibus caſibus cum difficile ſit numerum valorum involutorum conci-

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pere, unicus tantum realis conſideratur. Sic $a^{\sqrt{7}}$ erit valor CAP. VI.
determinatus intra limites a^2 & a^3 comprehenſus.

98. Maxime autem valores quantitatis exponentialis a^z a magnitudine numeri conſtantis a pendebunt. Si enim fuerit $a = 1$, ſemper erit $a^z = 1$, quicunque valores Exponenti z tribuantur; ſin autem fuerit $a^z > 1$, tum valor ipſius a^z eo erit major, quo major numerus loco z ſubſtituatur, atque adeo, poſito $z = \infty$, a^z in infinitum excreſcit; ſi fuerit $z = 0$, fiet $a^z = 1$, &, ſi ſit $z < 0$ valores a^z fient unitate minores, quoad poſito $z = -\infty$ fiat $a^z = 0$. Contrarium evenit ſi ſit $a < 1$, verum tamen quantitas affirmativa; tum enim valores ipſius a^z decreſcent, creſcente z ſupra 0; creſcent vero, ſi pro z numeri negativi ſubſtituantur. Cum enim ſit $a < 1$, erit $\frac{1}{a} > 1$; poſito ergo $\frac{1}{a} = b$; erit $a^z = b^{-z}$, unde poſterior caſus ex priori dijudicari poterit.

99. Si ſit $a = 0$, ingens ſaltus in valoribus ipſius a^z deprehenditur, quamdiu enim fuerit z numerus affirmativus ſeu major nihilo, erit perpetuo $a^z = 0$: ſi ſit $z = 0$ erit $a^0 = 1$; ſin autem fuerit z numerus negativus, tum a^z obtinebit valorem infinite magnum. Sit enim $z = -3$; erit $a^z = 0^{-3} = \frac{1}{0^3} = \frac{1}{0}$, ideoque infinitum. Multo majores autem ſaltus occurrent, ſi quantitas conſtans a habeat valorem negativum, puta -2; tum enim ponendis loco z numeris integris valores ipſius a^z alternatim erunt affirmativi & negativi, ut ex hac Serie intelligitur

$$a^{-4};\ a^{-3};\ a^{-2};\ a^{-1};\ a^0;\ a^1;\ a^2;\ a^3;\ a^4;\ \&c.$$

$+\frac{1}{16}; -\frac{1}{8}; +\frac{1}{4}; -\frac{1}{2}; 1; -2; +4; -8; +16$. Præterea vero ſi Exponenti z valores tribuantur fracti, Poteſtas $a^z = (-2)^z$ mox reales mox imaginarios induet valores: erit enim $a^{\frac{1}{2}} = \sqrt{-2}$ imaginarium; at erit $a^{\frac{1}{3}} = \sqrt[3]{-2} = -\sqrt[3]{2}$ reale: utrum autem, ſi Exponenti z tribuantur valores irrationales, Poteſtas a^z exhibeat quantitates reales an imaginarias, ne quidem definiri licet.

100. His igitur incommodis numerorum negativorum loco a ſubſtituendorum commemoratis, ſtatuamus a eſſe numerum affirmativum, & unitate quidem majorem, quia huc quoque illi caſus, quibus a eſt numerus affirmativus unitate minor, facile reducuntur. Si ergo ponatur $a^z = y$, loco z ſubſtituendo omnes numeros reales, qui intra limites $+\infty$ & $-\infty$ continentur, y adipiſcetur omnes valores affirmativos intra limites $+\infty$ & 0 contentos. Si enim ſit $z = \infty$ erit $y = \infty$; ſi $z = 0$ erit $y = 1$, & ſi $z = -\infty$ fiet $y = 0$. Viciſſim ergo quicunque valor affirmativus pro y accipiatur, dabitur quoque valor realis reſpondens pro z ita ut ſit $a^z = y$; ſin autem ipſi y tribueretur valor negativus, Exponens z valorem realem habere non poterit.

101. Si igitur fuerit $y = a^z$, erit y Functio quædam ipſius z, &, quemadmodum y a z pendeat, ex natura Poteſtatum facile intelligitur; hinc enim quicunque valor ipſi z tribuatur, valor ipſius y determinatur. Erit autem $yy = a^{2z}$; $y^3 = a^{3z}$: & generaliter erit $y^n = a^{nz}$; unde ſequitur fore $\sqrt{y} = a^{\frac{1}{2}z}$; $\sqrt[3]{y} = a^{\frac{1}{3}z}$ & $\frac{1}{y} = a^{-z}$; $\frac{1}{yy} = a^{-2z}$; & $\frac{1}{\sqrt{y}} = a^{-\frac{1}{2}z}$, & ita porro. Præterea, ſi fuerit $v = a^x$ erit $vy = a^{x+z}$ & $\frac{v}{y} = a^{x-z}$, quorum ſubſidiorum beneficio eo facilius valor ipſius y ex dato valore ipſius z inveniri poteſt.

EXEMPLUM.

EXEMPLUM.

Si fuerit $a = 10$, ex Arithmetica, qua utimur, denaria in promtu erit valores ipſius y exhibere, ſi quidem pro z numeri integri ponantur. Erit enim $10^1 = 10$; $10^2 = 100$; $10^3 = 1000$; $10^4 = 10000$; & $10^0 = 1$; item $10^{-1} = \frac{1}{10} = 0,1$; $10^{-2} = \frac{1}{100} = 0,01$; $10^{-3} = \frac{1}{1000} = 0,001$: ſin autem pro z Fractiones ponantur, ope radicum extractionis valores ipſius y indicari poſſunt: ſic erit $10^{\frac{1}{2}} = \sqrt{10} = 3,162277$, &c.

102. Quemadmodum autem, dato numero a, ex quovis valore ipſius z reperiri poteſt valor ipſius y, ita viciſſim, dato valore quocunque affirmativo ipſius y, conveniens dabitur valor ipſius z, ut ſit $a^z = y$; iſte autem valor ipſius z, quatenus tanquam Functio ipſius y ſpectatur, vocari ſolet LOGARITHMUS ipſius y. Supponit ergo doctrina Logarithmorum numerum certum conſtantem loco a ſubſtituendum, qui propterea vocatur *baſis* Logarithmorum; qua aſſumta erit Logarithmus cujuſque numeri y Exponens Poteſtatis a^z, ita ut ipſa Poteſtas a^z æqualis ſit numero illi y; indicari autem Logarithmus numeri y ſolet hoc modo ly. Quod ſi ergo fuerit $a^z = y$, erit $z = ly$: ex quo intelligitur, baſin Logarithmorum, etiamſi ab arbitrio noſtro pendeat, tamen eſſe debere numerum unitate majorem. hincque nonniſi numerorum affirmativorum Logarithmos realiter exhiberi poſſe.

103. Quicunque ergo numerus pro baſi Logarithmica a accipiatur, erit ſemper $l1 = 0$; ſi enim in æquatione $a^z = y$, quæ convenit cum hac $z = ly$, ponatur $y = 1$, erit $z = 0$. Deinde numerorum unitate majorum Logarithmi erunt affirmativi, pendentes a valore baſis a, ſic erit $la = 1$; $laa = 2$; $la^3 = 3$;

$la^4 = 4$; &c., unde a poſteriori intelligi poteſt, quantus numerus pro baſi ſit aſſumtus, ſcilicet ille numerus, cujus Logarithmus eſt $= 1$, erit baſis Logarithmica. Numerorum autem unitate minorum, affirmativorum tamen, Logarithmi erunt negativi; erit enim $l\frac{1}{a} = -1$; $l\frac{1}{aa} = -2$; $l\frac{1}{a^3} = -3$, &c.; numerorum autem negativorum Logarithmi non erunt reales, ſed imaginarii, uti jam notavimus.

104. Simili modo ſi fuerit $ly = z$; erit $lyy = 2z$; $ly^3 = 3z$; & generaliter $ly^n = nz$, ſeu $ly^n = nly$, ob $z = ly$. Logarithmus igitur cujuſque Poteſtatis ipſius y æquatur Logarithmo ipſius y per Exponentem Poteſtatis multiplicato; ſic erit $l\sqrt{y} = \frac{1}{2}z = \frac{1}{2}ly$; $l\frac{1}{\sqrt{y}} = ly^{-\frac{1}{2}} = -\frac{1}{2}ly$; & ita porro: unde ex dato Logarithmo cujuſque numeri inveniri poſſunt Logarithmi quarumcunque ipſius Poteſtatum. Sin autem jam inventi ſint duo Logarithmi, nempe $ly = z$ & $lv = x$: cum ſit $y = a^z$ & $v = a^x$ erit $lvy = x + z = lv + ly$; hinc Logarithmus Producti duorum numerorum æquatur ſummæ Logarithmorum Factorum; ſimili vero modo erit $l\frac{y}{v} = z - x = ly - lv$; hincque Logarithmus Fractionis æquatur Logarithmo Numeratoris dempto Logarithmo Denominatoris: quæ regulæ inſerviunt Logarithmis plurium numerorum inveniendis, ex cognitis jam aliquot Logarithmis.

105. Ex his autem patet aliorum numerorum non dari Logarithmos rationales, niſi Poteſtatum baſeos a; niſi enim numerus alius b fuerit Poteſtas baſis a, ejus Logarithmus numero rationali exprimi non poterit. Neque vero etiam Logarithmus ipſius b erit numerus irrationalis; ſi enim foret $lb = \sqrt{n}$, tum eſſet $a^{\sqrt{n}} = b$; id quod fieri nequit, ſi quidem numeri a & b

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rationales ſtatuantur; ſolent autem imprimis numerorum rationalium & integrorum Logarithmi deſiderari, quia ex his Logarithmi Fractionum ac numerorum ſurdorum inveniri poſſunt. Cum igitur Logarithmi numerorum, qui non ſunt Poteſtates baſis a, neque rationaliter neque irrationaliter exhiberi queant, merito ad quantitates tranſcendentes referuntur, hincque Logarithmi quantitatibus tranſcendentibus annumerari ſolent.

106. Hanc ob rem Logarithmi numerorum vero tantum proxime per Fractiones decimales exprimi ſolent, qui eo minus à veritate diſcrepabunt, quod ad plures figuras fuerint exacti. Atque hoc modo per ſolam radicis quadratæ extractionem cujuſque numeri Logarithmus vero proxime determinari poterit. Cum enim, poſito $ly = z$ & $lv = x$, ſit $l\sqrt{vy} = \frac{x+z}{2}$; ſi numerus propoſitus b contineatur intra limites a^2 & a^3, quorum Logarithmi ſunt 2 & 3, quæratur valor ipſius $a^{2\frac{1}{2}}$ ſeu $a^2\sqrt{a}$, atque b vel intra limites a^2 & $a^{2\frac{1}{2}}$ vel $a^{2\frac{1}{2}}$ & a^3 continebitur, utrumvis accidat, ſumendo medio proportionali, denuo limites propiores prodibunt, hocque modo ad limites pervenire licebit, quorum intervallum data quantitate minus evadat, & quibuſcum numerus propoſitus b ſine errore confundi poſſit. Quoniam vero horum ſingulorum limitum Logarithmi dantur, tandem Logarithmus numeri b reperietur.

EXEMPLUM.

Ponatur baſis Logarithmica $a = 10$, quod in tabulis uſu receptis fieri ſolet; & quæratur vero tantum proxime Logarithmus numeri 5; quia hic continetur intra limites 1 & 10 quorum Logarithmi ſunt 0 & 1; ſequenti modo radicum extractio continua inſtituatur, quoad ad limites à numero propoſito 5 non amplius diſcrepantes perveniatur.

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$A = 1,000000$;	$lA = 0,0000000$;	ſit
$B = 10,000000$;	$lB = 1,0000000$;	$C = \sqrt{AB}$
$C = 3,162277$;	$lC = 0,5000000$;	$D = \sqrt{BC}$
$D = 5,623413$;	$lD = 0,7500000$;	$E = \sqrt{CD}$
$E = 4,216964$;	$lE = 0,6250000$;	$F = \sqrt{DE}$
$F = 4,869674$;	$lF = 0,6875000$;	$G = \sqrt{DF}$
$G = 5,232991$;	$lG = 0,7187500$;	$H = \sqrt{FG}$
$H = 5,048065$;	$lH = 0,7031250$;	$I = \sqrt{FH}$
$I = 4,958069$;	$lI = 0,6953125$;	$K = \sqrt{HI}$
$K = 5,002865$;	$lK = 0,6992187$;	$L = \sqrt{IK}$
$L = 4,980416$;	$lL = 0,6972656$;	$M = \sqrt{KL}$
$M = 4,991627$;	$lM = 0,6982421$;	$N = \sqrt{KM}$
$N = 4,997242$;	$lN = 0,6987304$;	$O = \sqrt{KN}$
$O = 5,000052$;	$lO = 0,6989745$;	$P = \sqrt{NO}$
$P = 4,998647$;	$lP = 0,6988525$;	$Q = \sqrt{OP}$
$Q = 4,999350$;	$lQ = 0,6989135$;	$R = \sqrt{OQ}$
$R = 4,999701$;	$lR = 0,6989440$;	$S = \sqrt{OR}$
$S = 4,999876$;	$lS = 0,6989592$;	$T = \sqrt{OS}$
$T = 4,999963$;	$lT = 0,6989668$;	$V = \sqrt{OT}$
$V = 5,000008$;	$lV = 0,6989707$;	$W = \sqrt{TV}$
$W = 4,999984$;	$lW = 0,6989687$;	$X = \sqrt{WV}$
$X = 4,999997$;	$lX = 0,6989697$;	$Y = \sqrt{VX}$
$Y = 5,000003$;	$lY = 0,6989702$;	$Z = \sqrt{XY}$
$Z = 5,000000$;	$lZ = 0,6989700$;	

Sic ergo mediis proportionalibus ſumendis tandem perventum eſt ad $Z = 5,000000$, ex quo Logarithmus numeri 5 quæſitus eſt $0,698970$, poſita baſi Logarithmica $= 10$. Quare erit proxime $10^{\frac{69897}{100000}} = 5$. Hoc autem modo computatus eſt canon Logarithmorum vulgaris à BRIGGIO & VLACQUIO, quamquam poſtea eximia inventa ſunt compendia, quorum ope multo expeditius Logarithmi ſupputari poſſunt.

107. Dantur ergo tot diverſa Logarithmorum ſyſtemata quot varii numeri pro baſi a accipi poſſunt, atque ideo numerus ſyſ-

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tematum Logarithmicorum erit infinitus. Perpetuo autem in duobus ſyſtematis Logarithmi ejuſdem numeri eandem inter ſe ſervant rationem. Sit baſis unius ſyſtematis $= a$, alterius $= b$, atque numeri n Logarithmus in priori ſyſtemate $= p$, in poſteriori $= q$; erit $a^p = n$ & $b^q = n$; unde $a^p = b^q$; ideoque $a = b^{\frac{q}{p}}$. Oportet ergo ut Fractio $\frac{q}{p}$ conſtantem obtineat valorem, quicunque numerus pro n fuerit aſſumtus. Quod ſi ergo pro uno ſyſtemate Logarithmi omnium numerorum fuerint computati, hinc facili negotio per regulam auream Logarithmi pro quovis alio ſyſtemate reperiri poſſunt. Sic, cum dentur Logarithmi pro baſi 10, hinc Logarithmi pro quavis alia baſi, puta 2, inveniri poſſunt; quæratur enim Logarithmus numeri n pro baſi 2, qui ſit $= q$, cum ejuſdem numeri n Logarithmus ſit $= p$ pro baſi 10. Quoniam pro baſi 10 eſt $l\,2 = 0,3010300$, & pro baſi 2, eſt $l\,2 = 1$, erit $0,3010300 : 1 = p : q$ ideoque $q = \frac{p}{0,3010300} = 3,3219277.\,p$; ſi ergo omnes Logarithmi communes multiplicentur per numerum $3,3219277$, prodibit tabula Logarithmorum pro baſi 2.

108. *Hinc ſequitur duorum numerorum Logarithmos in quocunque ſyſtemate eandem tenere rationem.*

Sint enim duo numeri M & N, quorum pro baſi a Logarithmi ſint m & n, erit $M = a^m$ & $N = a^n$: hinc fiet $a^{mn} = M^n = N^m$, ideoque $M = N^{\frac{m}{n}}$; in qua æquatione cum baſis a non amplius inſit, perſpicuum eſt Fractionem $\frac{m}{n}$ habere valorem à baſi a non pendentem. Sint enim pro alia baſi b numerorum eorundem M & N Logarithmi μ & ν, pari modo colligetur fore $M = N^{\frac{\mu}{\nu}}$. Erit ergo $N^{\frac{m}{n}} = N^{\frac{\mu}{\nu}}$, hincque $\frac{m}{n} = \frac{\mu}{\nu}$, ſeu $m : n = \mu : \nu$. Ita jam vidimus

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 in omni Logarithmorum ſyſtemate Logarithmos diverſarum ejuſdem numeri Poteſtatum ut y^m & y^n tenere rationem Exponentium $m : n$.

109. Ad canonem ergo Logarithmorum pro baſi quacunque a condendum ſufficit numerorum tantum primorum Logarithmos methodo ante tradita, vel alia commodiori, ſupputaſſe. Cum enim Logarithmi numerorum compoſitorum ſint æquales ſummis Logarithmorum ſingulorum Factorum, Logarithmi numerorum compoſitorum per ſolam additionem reperientur. Sic, ſi habeantur logarithmi numerorum 3 & 5, erit $l\,15 = l\,3 + l\,5$; $l\,45 = 2\,l\,3 + l\,5$. Atque, cum ſupra pro baſi $a = 10$, inventus ſit $l\,5 = 0,6989700$, præterea autem ſit $l\,10 = 1$, erit $l\,\frac{10}{5} = l\,2 = l\,10 - l\,5$, ideoque orietur $l\,2 = 1 - 0,6989700 = 0,3010300$; ex his autem numerorum primorum 2 & 5 Logarithmis inventis reperientur Logarithmi omnium numerorum ex his 2 & 5 compoſitorum; cujuſmodi ſunt iſti 4, 8, 16, 32, 64, &c; 20, 40, 80, 25, 50, &c.

110. Tabularum autem Logarithmicarum ampliſſimus eſt uſus in contrahendis calculis numericis, propterea quod ex ejuſmodi tabulis non ſolum dati cujuſque numeri Logarithmus, ſed etiam cujuſque Logarithmi propoſiti numerus conveniens reperiri poteſt. Sic, ſi c, d, e, f, g, h, denotent numeros quoſcunque, citra multiplicationem reperiri poterit valor iſtius expreſſionis $\frac{cc\,d\,\sqrt{e}}{f\,\sqrt[3]{g\,h}}$, erit enim hujus expreſſionis Logarithmus $= 2\,l\,c + l\,d + \frac{1}{2}\,l\,e - l\,f - \frac{1}{3}\,l\,g - \frac{1}{3}\,l\,h$, cui Logarithmo ſi quæratur numerus reſpondens, habebitur valor quæſitus. Inprimis autem inſerviunt tabulæ Logarithmicæ dignitatibus atque radicibus intricatiſſimis inveniendis, quarum operationum loco in Logarithmis tantum multiplicatio aut diviſio adhibetur.

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EXEMPLUM I.

Quæratur valor hujus Poteſtatis $2^{\frac{7}{12}}$: quoniam ejus Logarithmus eſt $= \frac{7}{12}\, l\, 2$, multiplicetur Logarithmus binarii ex tabulis qui eſt 0, 3010300 per $\frac{7}{12}$ hoc eſt per $\frac{1}{2} + \frac{1}{12}$ erit $l\, 2^{\frac{7}{12}} = 0,\ 1756008$, cui Logarithmo reſpondet numerus 1, 498307, qui ergo proxime exhibet valorem $2^{\frac{7}{12}}$.

EXEMPLUM II.

Si numerus incolarum cujuſpiam provinciæ quotannis ſui parte trigeſima augeatur, initio autem in provincia habitaverint 100000 hominum, quæritur poſt 100 annos incolarum numerus. Sit brevitatis gratia initio incolarum numerus $= n$, ita ut ſit $n =$ 100000, anno elapſo uno erit incolarum numerus $= (1 + \frac{1}{30})n$ $= \frac{31}{30}\, n$: poſt duos annos $= (\frac{31}{30})^2 n$: poſt tres annos $= (\frac{31}{30})^3 n$, hincque poſt centum annos $= (\frac{31}{30})^{100} n =$ $(\frac{31}{30})^{100}$ 100000; cujus Logarithmus eſt $= 100\, l \frac{31}{30} +$ l 100000. At eſt $l \frac{31}{30} = l\, 31 - l\, 30 = 0,\ 014240439$, unde $100\, l \frac{31}{30} = 1,\ 4240439$, ad quem ſi addatur l 100000 $= 5$, erit Logarithmus numeri incolarum quæſiti $= 6{,}4240439$, cui reſpondet numerus $= 2654874$. Poſt centum ergo annos numerus incolarum fit plus quam vicies ſexies cum ſemiſſe major.

LIB. I.

EXEMPLUM III.

Cum poſt diluvium a ſex hominibus genus humanum ſit propagatum, ſi ponamus ducentis annis poſt, numerum hominum jam ad 1000000 excreviſſe, quæritur quanta ſui parte numerus hominum quotannis augeri debuerit. Ponamus hoc tempore numerum hominum parte ſua $\frac{1}{x}$ quotannis increviſſe, atque poſt ducentos annos prodierit neceſſe eſt numerus hominum $= (\frac{1+x}{x})^{200} 6 = 1000000$, unde fit $\frac{1+x}{x} = (\frac{1000000}{6})^{\frac{1}{200}}$. Erit ergo $l\frac{1+x}{x} = \frac{1}{200} l \frac{1000000}{6} = \frac{1}{200}$. $5,2218487 = 0,0261092$, ideoque $\frac{1+x}{x} = \frac{1061963}{1000000}$, & $1000000 = 61963\, x$, unde fit $x = 16$ circiter. Ad tantam ergo hominum multiplicationem ſuffeciſſet, ſi quotannis decima ſexta ſui parte increviſſent; quæ multiplicatio ob longævam vitam non nimis magna cenſeri poteſt. Quod ſi autem eadem ratione per intervallum 400 annorum numerus hominum creſcere perrexiſſet, tum numerus hominum ad $1000000. \frac{1000000}{6} = 166666666666$ aſcendere debuiſſet, quibus ſuſtentandis univerſus orbis terrarum nequaquam par fuiſſet.

EXEMPLUM IV.

Si ſingulis ſeculis numerus hominum duplicetur, quæritur incrementum annuum. Si quotannis hominum numerum parte ſua $\frac{1}{x}$ creſcere ponamus, & initio numerus hominum fuerit $= n$, erit is poſt centum annos $= (\frac{1+x}{x})^{100} n$, qui cum eſſe debeat

beat $= 2n$, erit $\frac{1+x}{x} = 2^{\frac{1}{100}}$ & $l\frac{1+x}{x} = \frac{1}{100} l2 =$ $0,0030103$; hinc $\frac{1+x}{x} = \frac{10069555}{10000000}$; ergo $x =$ $\frac{10000000}{69555} = 144$, circiter; ſufficit ergo ſi numerus hominum quotannis parte ſua $\frac{1}{144}$ augeatur. Quam ob cauſam maxime ridiculæ ſunt eorum incredulorum hominum objectiones, qui negant tam brevi temporis ſpatio ab uno homine univerſam terram incolis impleri potuiſſe.

111. Potiſſimum autem Logarithmorum uſus requiritur ad ejuſmodi æquationes reſolvendas, in quibus quantitas incognita in Exponentem ingreditur. Sic, ſi ad hujuſmodi perveniatur æquationem $a^x = b$, ex qua incognitæ x valorem erui oporteat, hoc non niſi per Logarithmos effici poterit. Cum enim ſit $a^x = b$ erit $la^x = xla = lb$, ideoque $x = \frac{lb}{la}$, ubi quidem perinde eſt, quonam ſyſtemate Logarithmico utatur, cum in omni ſyſtemate Logarithmi numerorum a & b eandem inter ſe teneant rationem.

EXEMPLUM I.

Si numerus hominum quotannis centeſima ſui parte augeatur; quæritur poſt quot annos numerus hominum fiat decuplo major. Ponamus hoc evenire poſt x annos, & initio hominum numerum fuiſſe $= n$, erit is ergo elapſis x annis $=$ $(\frac{101}{100})^x n$, qui cum æqualis ſit $10n$, fiet $(\frac{101}{100})^x = 10$; ideoque $xl\frac{101}{100} = l10$ & $x = \frac{l10}{l101 - l100}$. Prodibit itaque $x = \frac{10000000}{43214} = 231$. Poſt annos ergo 231 fiet hominum

 numerus, quorum incrementum annuum tantum centeſimam partem efficit, decuplo major; hinc poſt 462 annos fiet centies, & poſt 693 annos millies major.

EXEMPLUM II.

Quidam debet 400000 florenos hac conditione ut quotannis uſuram 5 de centenis ſolvere teneatur; exſolvit autem ſingulis annis 25000 florenos: quæritur poſt quot annos debitum penitus extinguatur. Scribamus a pro debita ſumma 400000 fl. & b pro ſumma 25000 fl. quotannis ſoluta; debebit ergo elapſo uno anno $\frac{105}{100}\,a - b$; elapſis duobus annis $(\frac{105}{100})^2\,a - \frac{105}{100}\,b - b$; elapſis tribus annis $(\frac{105}{100})^3\,a - (\frac{105}{100})^2\,b - \frac{105}{100}\,b - b$; hinc, poſito brevitatis cauſa, n pro $\frac{105}{100}$, elapſis x annis adhuc debebit $n^x\,a - n^{x-1}\,b - n^{x-2}\,b - n^{x-3}\,b - \ldots\ldots\ldots - b = n^x\,a - b\,(1 + n + n^2 + \ldots. + n^{x-1})$. Cum igitur ſit ex natura progreſſionum geometricarum, $1 + n + n^2 + \ldots\ldots\ldots + n^{x-1} = \frac{n^x - 1}{n - 1}$, poſt x annos debitor adhuc debebit $n^x a - \frac{n^x\,b + b}{n - 1}$ flor., quod debitum nihilo æquale poſitum dabit hanc æquationem $n^x\,a = \frac{n^x\,b - b}{n - 1}$, ſeu $(n - 1)\,n^x\,a = n^x\,b - b$, ideoque $(b - na + a)\,n^x = b$ & $n^x = \frac{b}{b - (n - 1)\,a}$, unde fit $x = \frac{lb - l(b - (n - 1)\,a)}{ln}$. Cum jam ſit $a = 400000$, $b = 25000$, $n = \frac{105}{100}$, erit $(n - 1)\,a = 20000$ & $b - (n - 1)\,a = 5000$, atque annorum, quibus debitum penitus extinguitur, numerus $x =$

$\frac{l\,25000 - l\,5000}{l\frac{105}{100}} = \frac{l\,5}{l\frac{21}{20}} = \frac{6989700}{211893}$; erit ergo x aliquanto minor quam 33; ſcilicet elapſis annis 33 non ſolum debitum extinguetur, ſed creditor debitori reddere tenebitur $\frac{(n^{33} - 1)b}{n - 1}$ CAP. VI.

$- n^{33}a = \frac{(\frac{21}{20})^{33} \cdot 5000 - 25000}{\frac{1}{20}} = 100000 (\frac{21}{20})^{33} - 500000$

flor. Quia vero eſt $l\frac{21}{20} = 0,0211892991$, erit $l(\frac{21}{20})^{33} = 0,69924687$, & $l\,100000 (\frac{21}{20})^{33} = 5,6992469$, cui reſpondet hic numerus 500318, 8; unde creditor debitori poſt 33 annos reſtituere debet $318\frac{4}{5}$ florenos.

112. Logarithmi autem vulgares ſuper baſi $= 10$ inſtructi, præter hunc uſum, quem Logarithmi in genere præſtant, in Arithmetica decimali uſu recepta ſingulari gaudent commodo, atque ob hanc cauſam præ aliis ſyſtematibus inſignem afferunt utilitatem. Cum enim Logarithmi omnium numerorum, præter denarii Poteſtates, in Fractionibus decimalibus exhibeantur, numerorum inter 1 & 10 contentorum Logarithmi intra limites 0 & 1, numerorum autem inter 10 & 100 contentorum Logarithmi inter limites 1 & 2, & ita porro, continebuntur. Conſtat ergo Logarithmus quiſque ex numero integro & Fractione decimali; & ille numerus integer vocari ſolet CHARACTERISTICA; Fractio decimalis autem MANTISSA. Characteriſtica itaque unitate deficiet a numero notarum, quibus numerus conſtat; ita Logarithmi numeri 78509 Characteriſtica erit 4, quia is ex quinque notis ſeu figuris conſtat. Hinc ex Logarithmo cujuſvis numeri ſtatim intelligitur, ex quot figuris numerus ſit compoſitus. Sic numerus Logarithmo 7,5804631 reſpondens ex 8 figuris conſtabit.

113. Si ergo duorum Logarithmorum Mantiſſæ conveniant, Characteriſticæ vero tantum diſcrepent, tum numeri his Loga-

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LIB. I. rithmis reſpondentes rationem habebunt, ut Poteſtas denarii ad unitatem, ideoque ratione figurarum, quibus conſtant, convenient. Ita horum Logarithmorum, 4, 9130187 & 6, 9130187 numeri erunt 81850 & 8185000; Logarithmo autem 3, 9130187 conveniet 8185, & Logarithmo huic 0, 9130187 convenit 8, 185. Sola ergo Mantiſſa indicabit figuras numerum componentes, quibus inventis, ex Characteriſtica patebit, quot figuræ a ſiniſtra ad integra referri debeant, reliquæ ad dextram vero dabunt Fractiones decimales. Sic, ſi hic Logarithmus fuerit inventus 2, 7603429, Mantiſſa indicabit has figuras 5758945, Characteriſtica 2 autem numerum illi Logarithmo reſpondentem determinat, ut ſit 575, 8945; ſi Characteriſtica eſſet 0, foret numerus 5, 758945; ſin denuo unitate minuatur ut ſit — 1, erit numerus reſpondens decies minor, nempe 0, 5758945; & Characteriſticæ — 2 reſpondebit 0, 05758945 &c.: loco Characteriſticarum autem hujuſmodi negativarum — 1, — 2, — 3, &c. ſcribi ſolent 9, 8, 7, &c., atque ſubintelligitur hos Logarithmos denario minui debere. Hæc vero in manuductionibus ad tabulas Logarithmorum fuſius exponi ſolent.

EXEMPLUM.

Si hæc progreſſio 2, 4, 16, 256, &c., *cujus quiſque terminus eſt quadratum præcedentis, continuetur uſque ad terminum vigeſimum quintum; quæritur magnitudo hujus termini ultimi.* Termini hujus progreſſionis per Exponentes ita commodius exprimuntur: $2^1, 2^2, 2^4, 2^8$, &c. ubi patet Exponentes progreſſionem geometricam conſtituere, atque termini vigeſimi quinti exponentem fore $2^{24} = 16777216$, ita ut ipſe terminus quæſitus ſit $= 2^{16777216}$, hujus ergo Logarithmus erit $= 16777216.\,l2$. Cum ergo ſit $l2 = 0, 30102999566398 1195$, erit numeri quæſiti Logarithmus $= 5050445, 25973367$, ex cujus Characteriſtica patet numerum quæſitum more ſolito expreſſum conſtare

ex 5050446 figuris. Mantiſſa autem 259733675932 in tabula Logarithmorum quæſita dabit figuras initiales numeri quæſiti, quæ erunt 181858. Quanquam ergo iſte numerus nullo modo exhiberi queat, tamen affirmari poteſt eum omnino ex 5050446 figuris conſtare, atque figuras initiales ſex eſſe 181858, quas dextrorſum adhuc 5050440 figuræ ſequantur, quarum inſuper nonnullæ ex majori Logarithmorum canone definiri poſſent, undecim ſcilicet figuræ initiales erunt 18185852986.

CAPUT VII.

De quantitatum exponentialium ac Logarithmorum per Series explicatione.

114. Quia eſt $a^0 = 1$, atque creſcente Exponente ipſius a ſimul valor Poteſtatis augetur, ſi quidem a eſt numerus unitate major; ſequitur ſi Exponens infinite parum cyphram excedat, Poteſtatem ipſam quoque infinite parum unitatem eſſe ſuperaturam. Sit ω numerus infinite parvus, ſeu Fractio tam exigua, ut tantum non nihilo ſit æqualis, erit $a^\omega = 1 + \psi$, exiſtente ψ quoque numero infinite parvo. Ex præcedente enim capite conſtat niſi ψ eſſet numerus infinite parvus, neque ω talem eſſe poſſe. Erit ergo vel $\psi = \omega$, vel $\psi > \omega$, vel $\psi < \omega$, quæ ratio utique a quantitate litteræ a pendebit, quæ cum adhuc ſit incognita, ponatur $\psi = k\omega$, ita ut ſit $a^\omega = 1 + k\omega$; &, ſumta a pro baſi Logarithmica, erit $\omega = l(1 + k\omega)$.

EXEMPLUM.

Quo clarius appareat, quemadmodum numerus k pendeat a baſi a, ponamus eſſe $a = 10$; atque ex tabulis vulgaribus quæramus Logarithmum numeri quam minime unitatem ſupe-

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 rantis, puta $1 + \frac{1}{1000000}$, ita ut ſit $k\omega = \frac{1}{1000000}$; erit $l(1 + \frac{1}{1000000}) = l\frac{1000001}{1000000} = 0,00000043429 = \omega$. Hinc, ob $k\omega = 0,00000100000$, erit $\frac{1}{k} = \frac{43429}{100000}$ & $k = \frac{100000}{43429} = 2,30258$: unde patet k eſſe numerum finitum pendentem a valore baſis a. Si enim alius numerus pro baſi a ſtatuatur, tum Logarithmus ejuſdem numeri $1 + k\omega$ ad priorem datam tenebit rationem, unde ſimul alius valor litteræ k prodiret.

115. Cum ſit $a^\omega = 1 + k\omega$, erit $a^{i\omega} = (1 + k\omega)^i$, quicunque numerus loco i ſubſtituatur. Erit ergo $a^{i\omega} = 1 + \frac{i}{1}k\omega + \frac{i(i-1)}{1.\ 2}k^2\omega^2 + \frac{i(i-1)(i-2)}{1.\ 2.\ 3}k^3\omega^3 +$ &c. Quod ſi ergo ſtatuatur $i = \frac{z}{\omega}$, & z denotet numerum quemcunque finitum, ob ω numerum infinite parvum, fiet i numerus infinite magnus, hincque $\omega = \frac{z}{i}$, ita ut ſit ω Fractio denominatorem habens infinitum, adeoque infinite parva, qualis eſt aſſumta. Subſtituatur ergo $\frac{z}{i}$ loco ω, eritque $a^z = (1 + \frac{kz}{i})^i = 1 + \frac{1}{1}kz + \frac{1(i-1)}{1.\ 2i}k^2z^2 + \frac{1(i-1)(i-2)}{1.\ 2i.\ 3i}k^3z^3 + \frac{1(i-1)(i-2)(i-3)}{1.\ 2i.\ 3i.\ 4i.}k^4z^4 +$ &c., quæ æquatio erit vera ſi pro i numerus infinite magnus ſubſtituatur. Tum vero eſt k numerus definitus ab a pendens, uti modo vidimus.

116. Cum autem i ſit numerus infinite magnus, erit $\frac{i-1}{i} = 1$; patet enim quo major numerus loco i ſubſtituatur, eo propius valorem Fractionis $\frac{i-1}{i}$ ad unitatem eſſe acceſſurum, hinc ſi

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i ſit numerus omni aſſignabili major, Fractio quoque $\frac{i-1}{i}$ ipſam unitatem adæquabit. Ob ſimilem autem rationem erit $\frac{i-2}{i} = 1$; $\frac{i-3}{i} = 1$; & ita porro; hinc ſequitur fore $\frac{i-1}{2i} = \frac{1}{2}$; $\frac{i-2}{3i} = \frac{1}{3}$; $\frac{i-3}{4i} = \frac{1}{4}$; & ita porro. His igitur valoribus ſubſtitutis, erit $a^z = 1 + \frac{kz}{1} + \frac{k^2z^2}{1.2} + \frac{k^3z^3}{1.2.3} + \frac{k^4z^4}{1.2.3.4} +$ &c. in infinitum. Hæc autem æquatio ſimul relationem inter numeros a & k oſtendit, poſito enim $z = 1$, erit $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \frac{k^4}{1.2.3.4} +$ &c., atque ut a ſit $= 10$, neceſſe eſt ut ſit circiter $k = 2,30258$, uti ante invenimus.

117. Ponamus eſſe $b = a^n$, erit, ſumto numero a pro baſi Logarithmica, $lb = n$. Hinc, cum ſit $b^z = a^{nz}$, erit per Seriem infinitam $b^z = 1 + \frac{knz}{1} + \frac{k^2n^2z^2}{1.2} + \frac{k^3n^3z^3}{1.2.3} + \frac{k^4n^4z^4}{1.2.3.4} +$ &c., poſito vero lb pro n, erit $b^z = 1 + \frac{kz}{1} lb + \frac{k^2z^2}{1.2} (lb)^2 + \frac{k^3z^3}{1.2.3} (lb)^3 + \frac{k^4z^4}{1.2.3.4} (lb)^4 +$ &c.. Cognito ergo valore litteræ k ex dato valore baſis a, quantitas exponentialis quæcunque b^z per Seriem infinitam exprimi poterit, cujus termini ſecundum Poteſtates ipſius z procedant. His expoſitis oſtendamus quoque quomodo Logarithmi per Series infinitas explicari poſſint.

118. Cum ſit $a^\omega = 1 + k\omega$, exiſtente ω Fractione infinite parva, atque ratio inter a & k definiatur per hanc æquationem $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} +$ &c., ſi a ſumatur pro baſi Logarithmica, erit $\omega = l(1 + k\omega)$ & $i\omega = l(1 + k\omega)^i$.

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LIB. I. Manifeſtum autem eſt, quo major numerus pro i ſumatur, eo magis Poteſtatem $(1+k\omega)^i$ unitatem eſſe ſuperaturam; atque ſtatuendo $i =$ numero infinito, valorem Poteſtatis $(1+k\omega)^i$ ad quemvis numerum unitate majorem aſcendere. Quod ſi ergo ponatur $(1+k\omega)^i = 1+x$, erit $l(1+x) = i\omega$, unde, cum ſit $i\omega$ numerus finitus, Logarithmus ſcilicet numeri $1+x$, perſpicuum eſt, i eſſe debere numerum infinite magnum, alioquin enim $i\omega$ valorem finitum habere non poſſet.

119. Cum autem poſitum ſit $(1+k\omega)^i = 1+x$, erit $1+k\omega = (1+x)^{\frac{1}{i}}$ & $k\omega = (1+x)^{\frac{1}{i}} - 1$, unde fit $i\omega = \frac{i}{k}((1+x)^{\frac{1}{i}} - 1)$. Quia vero eſt $i\omega = l(1+x)$, erit $l(1+x) = \frac{i}{k}(1+x)^{\frac{1}{i}} - \frac{i}{k}$, poſito i numero infinite magno. Eſt autem $(1+x)^{\frac{1}{i}} = 1 + \frac{1}{i}x - \frac{1(i-1)}{i.\ 2i}x^2 + \frac{1(i-1)(2i-1)}{i.\ 2i.\ 3i}x^3 - \frac{1(i-1)(2i-1)(3i-1)}{i.\ 2i.\ 3i.\ 4i}x^4 +$ &c. Ob i autem numerum infinitum, erit $\frac{i-1}{2i} = \frac{1}{2}$; $\frac{2i-1}{3i} = \frac{2}{3}$; $\frac{3i-1}{4i} = \frac{3}{4}$, &c.; hinc erit $i(1+x)^{\frac{1}{i}} = i + \frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} +$ &c., & conſequenter $l(1+x) = \frac{1}{k}(\frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} +$ &c.$)$, poſita baſi Logarithmica $= a$ ac denotante k numerum huic baſi convenientem, ut ſcilicet ſit $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} +$ &c.

120. Cum igitur habeamus Seriem Logarithmo numeri $1+x$ æqualem, ejus ope ex data baſi a definire poterimus valorem numeri

numeri k. Si enim ponamus $1+x=a$, ob $la=1$, erit $1=\frac{1}{k}\left(\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\&c.\right)$, hincque habebitur $k=\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\&c.$, cujus ideo Seriei infinitæ valor, ſi ponatur $a=10$, circiter eſſe debebit $=2,30258$; quanquam difficulter intelligi poteſt eſſe $2,30258=\frac{9}{1}-\frac{9^2}{2}+\frac{9^3}{3}-\frac{9^4}{4}+\&c.$, quoniam hujus Seriei termini continuo fiunt majores, neque adeo aliquot terminis ſumendis ſumma vero propinqua haberi poteſt: cui incommodo mox remedium afferetur.

121. Quoniam igitur eſt $l(1+x)=\frac{1}{k}\left(\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\&c.\right)$, erit, poſito x negativo, $l(1-x)=-\frac{1}{k}\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\&c.\right)$. Subtrahatur Series poſterior a priori, erit $l(1+x)-l(1-x)=l\frac{1+x}{1-x}=\frac{2}{k}\times\left(\frac{x}{1}+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\&c.\right)$. Nunc ponatur $\frac{1+x}{1-x}=a$, ut ſit $x=\frac{a-1}{a+1}$, ob $la=1$ erit $k=2\left(\frac{a-1}{a+1}+\frac{(a-1)^3}{3(a+1)^3}+\frac{(a-1)^5}{5(a+1)^5}+\&c.\right)$, ex qua æquatione valor numeri k ex baſi a inveniri poterit. Si ergo baſis a ponatur $=10$ erit $k=2\left(\frac{9}{11}+\frac{9^3}{3.11^3}+\frac{9^5}{5.11^5}+\frac{9^7}{7.11^7}+\&c.\right)$, cujus Seriei termini ſenſibiliter decreſcunt, ideoque mox valorem pro k ſatis propinquum exhibent.

122. Quoniam ad ſyſtema Logarithmorum condendum baſin a pro lubitu accipere licet, ea ita aſſumi poterit ut fiat $k=1$. Ponamus ergo eſſe $k=1$, eritque per Seriem ſupra

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LIB. I. (116) inventam, $a = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$, qui termini, ſi in fractiones decimales convertantur atque actu addantur, præbebunt hunc valorem pro $a =$ $2,71828182845904523536028$, cujus ultima adhuc nota veritati eſt conſentanea. Quod ſi jam ex hac baſi Logarithmi conſtruantur, ii vocari ſolent Logarithmi *naturales* ſeu *hyperbolici*, quoniam quadratura hyperbolæ per iſtiuſmodi Logarithmos exprimi poteſt. Ponamus autem brevitatis gratia pro numero hoc $2,718281828459$ &c. conſtanter litteram e, quæ ergo denotabit baſin Logarithmorum naturalium ſeu hyperbolicorum, cui reſpondet valor litteræ $k = 1$; ſive hæc littera e quoque exprimet ſummam hujus Seriei $1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$ in infinitum.

123. Logarithmi ergo hyperbolici hanc habebunt proprietatem, ut numeri $1 + \omega$ Logarithmus ſit $= \omega$, denotante ω quantitatem infinite parvam, atque cum ex hac proprietate valor $k = 1$ innoteſcat, omnium numerorum Logarithmi hyperbolici exhiberi poterunt. Erit ergo, poſita e pro numero ſupra invento, perpetuo $e^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \&c.$ ipſi vero Logarithmi hyperbolici ex his Seriebus invenientur, quibus eſt $l(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c.$, & $l\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \&c.$, quæ Series vehementer convergunt, ſi pro x ſtatuatur fractio valde parva: ita ex Serie poſteriori facili negotio inveniuntur Logarithmi numerorum unitate non multo majorum. Poſito namque $x = \frac{1}{5}$, erit $l\frac{6}{4} = l\frac{3}{2} = \frac{2}{1.5} + \frac{2}{3.5^3} + \frac{2}{5.5^5} + \frac{2}{7.5^7} + \&c.$, & facto $x = \frac{1}{7}$, erit $l\frac{4}{3} = \frac{2}{1.7} + \frac{2}{3.7^3} + \frac{2}{5.7^5} +$

$\frac{2}{7 \cdot 7^7} +$ &c., facto $x = \frac{1}{9}$, erit $l\frac{5}{4} = \frac{2}{1 \cdot 9} + \frac{2}{3 \cdot 9^3} + \frac{2}{5 \cdot 9^5} + \frac{2}{7 \cdot 9^7} +$ &c.. Ex Logarithmis vero harum fractionum reperientur Logarithmi numerorum integrorum, erit enim ex natura Logarithmorum $l\frac{3}{2} + l\frac{4}{3} = l2$; tum $l\frac{3}{2} + l2 = l3$; & $2l2 = l4$; porro $l\frac{5}{4} + l4 = l5$; $l2 + l3 = l6$; $3l2 = l8$; $2l3 = l9$; & $l2 + l5 = l10$.

Exemplum.

Hinc Logarithmi hyperbolici numerorum ab 1 uſque ad 10 ita ſe habebunt, ut ſit

$l1$	=	0,	00000	00000	00000	00000	00000
$l2$	=	0,	69314	71805	59945	30941	72321
$l3$	=	1,	09861	22886	68109	69139	52452
$l4$	=	1,	38629	43611	19890	61883	44642
$l5$	=	1,	60943	79124	34100	37460	07593
$l6$	=	1,	79175	94692	28055	00081	24773
$l7$	=	1,	94591	01490	55313	30510	54639
$l8$	=	2,	07944	15416	79835	92825	16964
$l9$	=	2,	19722	45773	36219	38279	04905
$l10$	=	2,	30258	50929	94045	68401	79914

Hi ſcilicet Logarithmi omnes ex ſuperioribus tribus Seriebus ſunt deducti, præter $l7$, quem hoc compendio ſum aſſecutus. Poſui nimirum in Serie poſteriori $x = \frac{1}{99}$ ſicque obtinui $l\frac{100}{98} = l\frac{50}{49} = 0{,}02020270731751944840782 30$, qui ſubtractus a $l50 = 2l5 + l2 = 3{,}91202300542814605861875 08$, relinquit $l49$, cujus ſemiſſis dat $l7$.

LIB. I. 124. Ponatur Logarithmus hyperbolicus ipſius $1 + x$ ſeu $l(1 + x) = y$; erit $y = \frac{x}{1} - \frac{x}{2} + \frac{x^3}{3} - \frac{x^4}{4} +$ &c.. Sumto autem numero a pro baſi Logarithmica, ſit numeri ejuſdem $1 + x$ Logarithmus $= v$; erit, ut vidimus, $v = \frac{1}{k}(x - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} +$ &c.$) = \frac{y}{k}$; hincque $k = \frac{y}{v}$; ex quo commodiſſime valor ipſius k baſi a reſpondens ita definitur ut ſit æqualis cujuſvis numeri Logarithmo hyperbolico diviſo per Logarithmum ejuſdem numeri ex baſi a formatum. Poſito ergo numero hoc $= a$, erit $v = 1$, hincque fit $k =$ Logarithmo hyperbolico baſis a. In ſyſtemate ergo Logarithmorum communium, ubi eſt $a = 10$, erit $k =$ Logarithmo hyperbolico ipſius 10, unde fit $k = 2, 30258509299404568401799 14$, quem valorem jam ſupra ſatis prope collegimus. Si ergo ſinguli Logarithmi hyperbolici per hunc numerum k dividantur, vel, quod eodem redit, multiplicentur per hanc fractionem decimalem 0, 43429448190325182765 11289, prodibunt Logarithmi vulgares baſi $a = 10$ convenientes.

125. Cum ſit $e^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} +$ &c., ſi ponatur $a^y = e^z$, erit, ſumtis Logarithmis hyperbolicis, $y\,la = z$, quia eſt $le = 1$, quo valore loco z ſubſtituto, erit $a^y = 1 + \frac{y\,la}{1} + \frac{y^2(la)^2}{1.2} + \frac{y^3(la)^3}{1.2.3} +$ &c., unde quælibet quantitas exponentialis ope Logarithmorum hyperbolicorum per Seriem infinitam explicari poteſt. Tum vero, denotante i numerum infinite magnum, tam quantitates exponentiales quam Logarithmi per poteſtates exponi poſſunt. Erit enim $e^z = (1 + \frac{z}{i})^i$, hincque $ay = (1 + \frac{y\,la}{i})^i$, deinde pro Logarithmis hyperbolicis habetur $l(1 + x) = i((1 + x)^{\frac{1}{i}} - 1)$. De cetero

Logarithmorum hyperbolicorum uſus in calculo integrali fuſius demonſtrabitur.

CAPUT VIII.

De quantitatibus tranſcendentibus ex Circulo ortis.

126. Post Logarithmos & quantitates exponentiales conſiderari debent Arcus circulares eorumque Sinus & Coſinus, quia non ſolum aliud quantitatum tranſcendentium genus conſtituunt, ſed etiam ex ipſis Logarithmis & exponentialibus, quando imaginariis quantitatibus involvuntur, proveniunt, id quod infra clarius patebit.

Ponamus ergo Radium Circuli ſeu Sinum totum eſſe $= 1$, atque ſatis liquet Peripheriam hujus Circuli in numeris rationalibus exacte exprimi non poſſe, per approximationes autem inventa eſt Semicircumferentia hujus Circuli eſſe $=$ 3, 14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651327230664709384460 +, pro quo numero, brevitatis ergo, ſcribam π, ita ut ſit $\pi =$ Semicircumferentiæ Circuli, cujus Radius $= 1$, ſeu π erit longitudo Arcus 180 graduum.

127. Denotante z Arcum hujus Circuli quemcunque, cujus Radium perpetuo aſſumo $= 1$; hujus Arcus z conſiderari potiſſimum ſolent Sinus & Coſinus. Sinum autem Arcus z in poſterum hoc modo indicabo *ſin.* $A. z$, ſeu tantum *ſin.* z. Coſinum vero hoc modo *coſ.* $A. z$, ſeu tantum *coſ.* z. Ita, cum π ſit Arcus 180°, erit $\text{ſin.}\, 0\pi = 0$, $\text{coſ.}\, 0\pi = 1$; & $\text{ſin.}\, \frac{1}{2}\pi = 1$, $\text{coſ.}\, \frac{1}{2}\pi = 0$; $\text{ſin.}\, \pi = 0$, $\text{coſ.}\, \pi = -1$; $\text{ſin.}\, \frac{3}{2}\pi = -1$, $\text{coſ.}\, \frac{3}{2}\pi = 0$; $\text{ſin.}\, 2\pi = 0$ & $\text{coſ.}\, 2\pi = 1$.
Omnes ergo Sinus & Coſinus intra limites $+ 1$ & $- 1$ con-

LIB. I. tinentur. Erit autem porro $\text{coſ.}\, z = \text{ſin.}\, (\frac{1}{2}\pi - z)$, & $\text{ſin.}\, z = \text{coſ.}\, (\frac{1}{2}\pi - z)$, atque $(\text{ſin.}\, z)^2 + (\text{coſ.}\, z)^2 = 1$. Præter has denominationes notandæ ſunt quoque hæ: *tang.* z, quæ denotat Tangentem Arcus z; *cot.* z Cotangentem Arcus z; conſtatque eſſe $\text{tang.}\, z = \frac{\text{ſin.}\, z}{\text{coſ.}\, z}$ & $\text{cot.}\, z = \frac{\text{coſ.}\, z}{\text{ſin.}\, z} = \frac{1}{\text{tang.}\, z}$; quæ omnia ex Trigonometria ſunt nota.

128. Hinc vero etiam conſtat ſi habeantur duo Arcus y & z, fore $\text{ſin.}\, (y+z) = \text{ſin.}\, y.\, \text{coſ.}\, z + \text{coſ.}\, y.\, \text{ſin.}\, z$ & $\text{coſ.}\, (y+z) = \text{coſ.}\, y.\, \text{coſ.}\, z - \text{ſin.}\, y.\, \text{ſin.}\, z$, itemque $\text{ſin.}\, (y-z) = \text{ſin.}\, y.\, \text{coſ.}\, z - \text{coſ.}\, y.\, \text{ſin.}\, z$ & $\text{coſ.}\, (y-z) = \text{coſ.}\, y.\, \text{coſ.}\, z + \text{ſin.}\, y.\, \text{ſin.}\, z$.

Hinc loco y ſubſtituendo Arcus $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, &c., erit

$\text{ſin.}\, (\frac{1}{2}\pi + z) = +\, \text{coſ.}\, z$	$\text{ſin.}\, (\frac{1}{2}\pi - z) = +\, \text{coſ.}\, z$
$\text{coſ.}\, (\frac{1}{2}\pi + z) = -\, \text{ſin.}\, z$	$\text{coſ.}\, (\frac{1}{2}\pi - z) = +\, \text{ſin.}\, z$
$\text{ſin.}\, (\pi + z) = -\, \text{ſin.}\, z$	$\text{ſin.}\, (\pi - z) = +\, \text{ſin.}\, z$
$\text{coſ.}\, (\pi + z) = -\, \text{coſ.}\, z$	$\text{coſ.}\, (\pi - z) = -\, \text{coſ.}\, z$
$\text{ſin.}\, (\frac{3}{2}\pi + z) = -\, \text{coſ.}\, z$	$\text{ſin.}\, (\frac{3}{2}\pi - z) = -\, \text{coſ.}\, z$
$\text{coſ.}\, (\frac{3}{2}\pi + z) = +\, \text{ſin.}\, z$	$\text{coſ.}\, (\frac{3}{2}\pi - z) = -\, \text{ſin.}\, z$
$\text{ſin.}\, (2\pi + z) = +\, \text{ſin.}\, z$	$\text{ſin.}\, (2\pi - z) = -\, \text{ſin.}\, z$
$\text{coſ.}\, (2\pi + z) = +\, \text{coſ.}\, z$	$\text{coſ.}\, (2\pi - z) = +\, \text{coſ.}\, z$

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Si ergo n denotet numerum integrum quemcunque, erit

$\text{ſin.}\left(\frac{4n+1}{2}\pi+z\right)=+\text{coſ.}z$	$\text{ſin.}\left(\frac{4n+1}{2}\pi-z\right)=+\text{coſ.}z$
$\text{coſ.}\left(\frac{4n+1}{2}\pi+z\right)=-\text{ſin.}z$	$\text{coſ.}\left(\frac{4n+1}{2}\pi-z\right)=-\text{ſin.}z$
$\text{ſin.}\left(\frac{4n+2}{2}\pi+z\right)=-\text{ſin.}z$	$\text{ſin.}\left(\frac{4n+2}{2}\pi-z\right)=+\text{ſin.}z$
$\text{coſ.}\left(\frac{4n+2}{2}\pi+z\right)=-\text{coſ.}z$	$\text{coſ.}\left(\frac{4n+2}{2}\pi-z\right)=-\text{coſ.}z$
$\text{ſin.}\left(\frac{4n+3}{2}\pi+z\right)=-\text{coſ.}z$	$\text{ſin.}\left(\frac{4n+3}{2}\pi-z\right)=-\text{coſ.}z$
$\text{coſ.}\left(\frac{4n+3}{2}\pi+z\right)=+\text{ſin.}z$	$\text{coſ.}\left(\frac{4n+3}{2}\pi-z\right)=-\text{ſin.}z$
$\text{ſin.}\left(\frac{4n+4}{2}\pi+z\right)=+\text{ſin.}z$	$\text{ſin.}\left(\frac{4n+4}{2}\pi-z\right)=-\text{ſin.}z$
$\text{coſ.}\left(\frac{4n+4}{2}\pi+z\right)=+\text{coſ.}z$	$\text{coſ.}\left(\frac{4n+4}{2}\pi-z\right)=+\text{coſ.}z$

Quæ formulæ veræ ſunt ſive n ſit numerus affirmativus ſive negativus integer.

129. Sit $\text{ſin.}\,z=p$ & $\text{coſ.}\,z=q$, erit $pp+qq=1$; & $\text{ſin.}\,y=m$ & $\text{coſ.}\,y=n$, ut ſit quoque $mm+nn=1$; Arcuum ex his compoſitorum Sinus & Coſinus ita ſe habebunt.

$\text{ſin.}\ z=p$	$\text{coſ.}\ z=q$
$\text{ſin.}\,(y+z)=mq+np$	$\text{coſ.}\,(y+z)=nq-mp$
$\text{ſin.}\,(2y+z)=2mnq+(nn-mm)p$	$\text{coſ.}\,(2y+z)=(nn-mm)q-2mnp$
$\text{ſin.}\,(3y+z)=(3mn^2-m^3)q+(n^3-3m^2n)p$	$\text{coſ.}\,(3y+z)=(n^3-3m^2n)q-(3mn^2-m^3)p$
&c.	&c.

Arcus iſti z, $y+z$, $2y+z$, $3y+z$, &c., in arithmetica progreſſione progrediuntur; eorum vero tam Sinus quam Coſinus progreſſionem recurrentem conſtituunt, qualis ex denominatore $1-2nx+(mm+nn)xx$ oritur; eſt enim

 $\mathit{ſin.}\,(2y+z)=2n.\mathit{ſin.}\,(y+z)-(mm+nn)\,\mathit{ſin.}\,z$ ſive $\mathit{ſin.}\,(2y+z)=2\mathit{coſ}.y.\mathit{ſin.}(y+z)-(\mathit{ſin.}\,z)$; atque ſimili modo $\mathit{coſ}.(2y+z)=2\mathit{coſ}.y.\mathit{coſ}.(y+z)-\mathit{coſ}.z$. Eodem modo erit porro $\mathit{ſin.}\,(3y+z)=2\mathit{coſ}.y.\mathit{ſin.}(2y+z)-\mathit{ſin.}\,(y+z)$, & $\mathit{coſ}.(3y+z)=2\mathit{coſ}.y.\mathit{coſ}.(2y+z)-\mathit{coſ}.(y+z)$, itemque $\mathit{ſin.}\,(4y+z)=2\mathit{coſ}.y.\mathit{ſin.}(3y+z)-\mathit{ſin.}\,(2y+z)$, & $\mathit{coſ}.(4y+z)=2\mathit{coſ}.y.\mathit{coſ}.(3y+z)-\mathit{coſ}.(2y+z)$, &c. Cujus legis beneficio Arcuum in progreſſione arithmetica progredientium tam Sinus quam Coſinus quouſque libuerit expedite formari poſſunt.

130. Cum ſit $\mathit{ſin.}\,(y+z)=\mathit{ſin.}\,y.\,\mathit{coſ}.z+\mathit{coſ}.y.\mathit{ſin.}\,z$, atque $\mathit{ſin.}\,(y-z)=\mathit{ſin.}\,y.\,\mathit{coſ}.\,z-\mathit{coſ}.\,y.\,\mathit{ſin.}\,z$, erit his expreſſionibus vel addendis vel ſubtrahendis:

$$\mathit{ſin.}\,y.\,\mathit{coſ}.\,z=\frac{\mathit{ſin.}\,(y+z)+\mathit{ſin.}\,(y-z)}{2}$$
$$\mathit{coſ}.\,y.\,\mathit{ſin.}\,z=\frac{\mathit{ſin.}\,(y+z)-\mathit{ſin.}\,(y-z)}{2}$$

Quia porro eſt $\mathit{coſ}.\,(y+z)=\mathit{coſ}.y.\,\mathit{coſ}.z-\mathit{ſin.}\,y.\,\mathit{ſin.}\,z$, atque $\mathit{coſ}.\,(y-z)=\mathit{coſ}.\,y.\,\mathit{coſ}.z+\mathit{ſin.}\,y.\mathit{ſin.}\,z$, erit pari modo

$$\mathit{coſ}.\,y.\,\mathit{coſ}.\,z.=\frac{\mathit{coſ}.\,(y-z)+\mathit{coſ}.\,(y+z)}{2}$$
$$\mathit{ſin.}\,y.\,\mathit{ſin.}\,z.=\frac{\mathit{coſ}.\,(y-z)-\mathit{coſ}.\,(y+z)}{2}.$$

Sit $y=z=\frac{1}{2}v$, erit ex his poſtremis formulis:

$$(\mathit{coſ}.\tfrac{1}{2}v)^2=\frac{1+\mathit{coſ}.v}{2}, \ \&\ \mathit{coſ}.\tfrac{1}{2}v=\sqrt{\frac{1+\mathit{coſ}.v}{2}}$$
$$(\mathit{ſin.}\,\tfrac{1}{2}v)^2=\frac{1-\mathit{coſ}.v}{2}, \ \&\ \mathit{ſin.}\,\tfrac{1}{2}v=\sqrt{\frac{1-\mathit{coſ}.v}{2}}$$

unde, ex dato Coſinu cujuſque anguli reperiuntur ejus ſemiſſis Sinus & Coſinus.

131. Ponatur Arcus $y+z=a$, & $y-z=b$, erit $y=\frac{a+b}{2}$ & $z=\frac{a-b}{2}$; quibus in ſuperioribus formulis ſubſtitutis

tutis, habebuntur hæ æquationes, tanquam totidem Theoremata.

$$ſin.\, a + ſin.\, b = 2\, ſin.\, \frac{a+b}{2}\, coſ.\, \frac{a-b}{2}$$
$$ſin.\, a - ſin.\, b = 2\, coſ.\, \frac{a+b}{2}\, ſin.\, \frac{a-b}{2}$$
$$coſ.\, a + coſ.\, b = 2\, coſ.\, \frac{a+b}{2}\, coſ.\, \frac{a-b}{2}$$
$$coſ.\, b - coſ.\, a = 2\, ſin.\, \frac{a+b}{2}\, ſin.\, \frac{a-b}{2}.$$

Ex his porro naſcuntur, ope diviſionis, hæc Theoremata

$$\frac{ſin.\, a + ſin.\, b}{ſin.\, a - ſin.\, b} = tang.\, \frac{a+b}{2}\, cot.\, \frac{a-b}{2} = \frac{tang.\, \frac{a+b}{2}}{tang.\, \frac{a-b}{2}}$$
$$\frac{ſin.\, a + ſin.\, b}{coſ.\, a + coſ.\, b} = tang.\, \frac{a+b}{2}$$
$$\frac{ſin.\, a + ſin.\, b}{coſ.\, b - coſ.\, a} = cot.\, \frac{a-b}{2}$$
$$\frac{ſin.\, a - ſin.\, b}{coſ.\, a + coſ.\, b} = tang.\, \frac{a-b}{2}$$
$$\frac{ſin.\, a - ſin.\, b}{coſ.\, b - coſ.\, a} = cot.\, \frac{a+b}{2}$$
$$\frac{coſ.\, a + coſ.\, b}{coſ.\, b - coſ.\, a} = cot.\, \frac{a+b}{2}.\, cot.\, \frac{a-b}{2}$$

Ex his denique deducuntur iſta Theoremata

$$\frac{ſin.\, a + ſin.\, b}{coſ.\, a + coſ.\, b} = \frac{coſ.\, b - coſ.\, a}{ſin.\, a - ſin.\, b};$$
$$\frac{ſin.\, a + ſin.\, b}{ſin.\, a - ſin.\, b} \times \frac{coſ.\, a + coſ.\, b}{coſ.\, b - coſ.\, a} = \left(cot.\, \frac{a-b}{2} \right)^2$$
$$\frac{ſin.\, a + ſin.\, b}{ſin.\, a - ſin.\, b} \times \frac{coſ.\, b - coſ.\, a}{coſ.\, a + coſ.\, b} = \left(tang.\, \frac{a+b}{2} \right)^2.$$

132. Cum ſit $(ſin.\, z)^2 + (coſ.\, z)^2 = 1$ erit, Factoribus ſumptis, $(coſ.\, z + \sqrt{-1}.\, ſin.\, z)\, (coſ.\, z - \sqrt{-1}.\, ſin.\, z) = 1$; qui Factores, etſi imaginarii, tamen ingentem præſtant uſum in Arcubus combinandis & multiplicandis. Quæratur enim productum horum Factorum $(coſ.\, z + \sqrt{-1}.\, ſin.\, z)(coſ.\, y + \sqrt{-1}.\, ſin.\, y)$ ac reperietur $coſ.\, y.\, coſ.\, z - ſin.\, y.\, ſin.\, z + (coſ.\, y.\, ſin.\, z + ſin.\, y.\, coſ.\, z)$

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 $\sqrt{-1}$. Cum autem ſit $\cos. y. \cos. z - \sin. y. \sin. z = \cos. (y+z)$ & $\cos. y. \sin. z + \sin. y. \cos. z = \sin. (y+z)$ erit hoc productum $(\cos. y + \sqrt{-1}. \sin. y)(\cos. z + \sqrt{-1}. \sin. z) = \cos. (y+z) + \sqrt{-1}. \sin. (y+z)$

& ſimili modo

$(\cos. y - \sqrt{-1}. \sin. y)(\cos. z - \sqrt{-1}. \sin. z) = \cos. (y+z) - \sqrt{-1}. \sin. (y+z)$

item

$(\cos. x \pm \sqrt{-1}. \sin. x)(\cos. y \pm \sqrt{-1}. \sin. y)(\cos. z \pm \sqrt{-1}. \sin. z) = \cos. (x+y+z) \pm \sqrt{-1}. \sin. (x+y+z)$.

133. Hinc itaque ſequitur fore $(\cos. z \pm \sqrt{-1}. \sin. z)^2 = \cos. 2z \pm \sqrt{-1}. \sin. 2z$, & $(\cos. z \pm \sqrt{-1}. \sin. z)^3 = \cos. 3z \pm \sqrt{-1}. \sin. 3z$;

ideoque generaliter erit $(\cos. z \pm \sqrt{-1}. \sin. z)^n = \cos. nz \pm \sqrt{-1}. \sin. nz$:

Unde, ob ſignorum ambiguitatem, erit

$$\cos. nz = \frac{(\cos. z + \sqrt{-1}. \sin. z)^n + (\cos. z - \sqrt{-1}. \sin. z)^n}{2} \text{ \&}$$

$$\sin. nz = \frac{(\cos. z + \sqrt{-1}. \sin. z)^n - (\cos. z - \sqrt{-1}. \sin. z)^n}{2\sqrt{-1}}$$

Evolutis ergo binomiis hiſce erit per Series:

$$\cos. nz = (\cos. z)^n - \frac{n(n-1)}{1.2}(\cos. z)^{n-2}(\sin. z)^2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}(\cos. z)^{n-4}(\sin. z)^4 - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.2.3.4.5.6}(\cos. z)^{n-6}(\sin. z)^6 + \text{\&c.}, \text{ \&}$$

$$\sin. nz = \frac{n}{1}(\cos. z)^{n-1}\sin. z - \frac{n(n-1)(n-2)}{1.2.3}(\cos. z)^{n-3}(\sin. z)^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}(\cos. z)^{n-5}(\sin. z)^5 - \text{\&c.}$$

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134. Sit Arcus z infinite parvus, erit *ſin.* $z = z$ & *coſ.* $z = 1$: ſit autem n numerus infinite magnus, ut ſit Arcus nz finitæ magnitudinis, puta, $nz = v$; ob *ſin.* $z = z = \frac{v}{n}$ erit *coſ.* $v = 1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3.4.5.6} + \&c.$, & *ſin.* $v = v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3.4.5.6.7} + \&c.$ Dato ergo Arcu v, ope harum Serierum ejus Sinus & Coſinus inveniri poterunt; quarum formularum uſus quo magis pateat, ponamus Arcum v eſſe ad quadrantem, ſeu ad arcum 90°, ut m ad n, ſeu eſſe $v = \frac{m}{n} . \frac{\pi}{2}$; Quia nunc valor ipſius π conſtat, ſi is ubique ſubſtituatur, prodibit CAP. VIII.

ſin. A. $\frac{m}{n}$ 90° =

$$+ \frac{m}{n} . \; 1,5707963267948966192313216916$$
$$- \frac{m^3}{n^3} . \; 0,6459640975062462536557565636$$
$$+ \frac{m^5}{n^5} . \; 0,0796926262461670451205055488$$
$$- \frac{m^7}{n^7} . \; 0,0046817541353186881006854632$$
$$+ \frac{m^9}{n^9} . \; 0,0001604411847873598218726605$$
$$- \frac{m^{11}}{n^{11}} . \; 0,0000035988432352120853404580$$
$$+ \frac{m^{13}}{n^{13}} . \; 0,0000000569217292196792681171$$
$$- \frac{m^{15}}{n^{15}} . \; 0,0000000006688035109811467224$$
$$+ \frac{m^{17}}{n^{17}} . \; 0,0000000000060669357311061950$$

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$$- \frac{m^{19}}{n^{19}} \cdot 0,0000000000000437706546731370$$
$$+ \frac{m^{21}}{n^{21}} \cdot 0,0000000000000002571422892856$$
$$- \frac{m^{23}}{n^{23}} \cdot 0,0000000000000000012538995403$$
$$+ \frac{m^{25}}{n^{25}} \cdot 0,0000000000000000000051564550$$
$$- \frac{m^{27}}{n^{27}} \cdot 0,0000000000000000000000181239$$
$$+ \frac{m^{29}}{n^{29}} \cdot 0,0000000000000000000000000549,$$

atque *coſ.* A. $\frac{m}{n}$ $90^{\circ}=$

$$+ \frac{m^{0}}{n^{0}} \cdot 1,0000000000000000000000000000$$
$$- \frac{m^{2}}{n^{2}} \cdot 1,2337005501361698273543113745$$
$$+ \frac{m^{4}}{n^{4}} \cdot 0,2536695079010480136365633659$$
$$- \frac{m^{6}}{n^{6}} \cdot 0,0208634807633529608730516364$$
$$+ \frac{m^{8}}{n^{8}} \cdot 0,0009192602748394265802417158$$
$$- \frac{m^{10}}{n^{10}} \cdot 0,0000252020423730606054810526$$
$$+ \frac{m^{12}}{n^{12}} \cdot 0,0000004710874778818171503665$$
$$- \frac{m^{14}}{n^{14}} \cdot 0,0000000063866030837918522408$$
$$+ \frac{m^{16}}{n^{16}} \cdot 0,0000000000656596311497947230$$
$$- \frac{m^{18}}{n^{18}} \cdot 0,0000000000005294400200734620$$
$$+ \frac{m^{20}}{n^{20}} \cdot 0,0000000000000034377391790981$$
$$- \frac{m^{22}}{n^{22}} \cdot 0,0000000000000000183599165212$$

$+ \frac{m^{24}}{n^{24}}.\ 0,0000000000000000000820675327$ CAP. VIII.

$- \frac{m^{26}}{n^{26}}.\ 0,0000000000000000000003115285$

$+ \frac{m^{28}}{n^{28}}.\ 0,0000000000000000000000010165$

$- \frac{m^{30}}{n^{30}}.\ 0,0000000000000000000000000026.$

Cum igitur ſufficiat Sinus & Coſinus angulorum uſque ad 45° noſſe, fractio $\frac{m}{n}$ ſemper minor erit quam $\frac{1}{2}$, hincque etiam ob Poteſtates fractionis $\frac{m}{n}$, Series exhibitæ maxime convergent, ita ut plerumque aliquot tantum termini ſufficiant, præcipue, ſi Sinus & Coſinus non ad tot figuras perducti deſiderentur.

135. Inventis Sinibus & Coſinibus inveniri quidem poſſunt Tangentes & Cotangentes, per analogias conſuetas, at quia in hujuſmodi ingentibus numeris multiplicatio & diviſio vehementer eſt moleſta, peculiari modo eas exprimere convenit. Erit ergo

$$tang.\ v = \frac{ſin.\ v}{coſ.\ v} = \frac{v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3\ldots7} + \&c.}{1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3\ldots6} + \&c.}$$

$$\&\ cot.\ v = \frac{coſ.\ v}{ſin.\ v} = \frac{1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3\ldots6} + \&c.}{v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3\ldots7} + \&c.}$$

 ſi jam ſit Arcus $v = \frac{m}{n} 90^0$ erit eodem modo quo ante

tang. A. $\frac{m}{n} 90^0 =$		*cot.* A. $\frac{m}{n} 90^0 =$	
$+ \frac{2mn}{nn-mm}$.	0, 6366197723675	$+ \frac{n}{m}$.	0, 6366197723675
$+ \frac{m}{n}$.	0, 2975567820597	$- \frac{4mn}{4nn-mm}$.	0, 3183098861837
$+ \frac{m^3}{n^3}$.	0, 0186886502773	$- \frac{m}{n}$.	0, 2052888894145
$+ \frac{m^5}{n^5}$.	0, 0018424752034	$- \frac{m^3}{n^3}$.	0, 0065510747882
$+ \frac{m^7}{n^7}$.	0, 0001975800714	$- \frac{m^5}{n^5}$.	0, 0003450292554
$+ \frac{m^9}{n^9}$.	0, 0000216977245	$- \frac{m^7}{n^7}$.	0, 0000202791060
$+ \frac{m^{11}}{n^{11}}$.	0, 0000024011370	$- \frac{m^9}{n^9}$.	0, 0000012366527
$+ \frac{m^{13}}{n^{13}}$.	0, 0000002664132	$- \frac{m^{11}}{n^{11}}$.	0, 0000000764959
$+ \frac{m^{15}}{n^{15}}$.	0, 0000000295864	$- \frac{m^{13}}{n^{13}}$.	0, 0000000047597
$+ \frac{m^{17}}{n^{17}}$.	0, 0000000032867	$- \frac{m^{15}}{n^{15}}$.	0, 0000000002969
$+ \frac{m^{19}}{n^{19}}$.	0, 0000000003651	$- \frac{m^{17}}{n^{17}}$.	0, 0000000000185
$+ \frac{m^{21}}{n^{21}}$.	0, 0000000000405	$- \frac{m^{19}}{n^{19}}$.	0, 0000000000011
$+ \frac{m^{23}}{n^{23}}$.	0, 0000000000045		
$+ \frac{m^{25}}{n^{25}}$.	0, 0000000000005		

quarum Serierum ratio infra fuſius exponetur.

136. Ex ſuperioribus quidem conſtat, ſi cogniti fuerint omnium angulorum ſemirecto minorum Sinus & Coſinus, inde ſimul omnium angulorum majorum Sinus & Coſinus haberi. Verum ſi tantum angulorum 30^0 minorum habeantur

Sinus & Coſinus, ex iis, per ſolam additionem & ſubtractionem, omnium angulorum majorum Sinus & Coſinus inveniri poſſunt. Cum enim ſit $\text{ſin.}\ 30^o = \frac{1}{2}$, erit, poſito $y = 30^o$ (ex §. 130) $\text{coſ.}\ z = \text{ſin.}\ (30 + z) + \text{ſin.}\ (30 - z)$; & $\text{ſin.}\ z = \text{coſ.}\ (30 - z) - \text{coſ.}\ (30 + z)$, ideoque ex Sinibus & Coſinibus angulorum z & $30 - z$, reperiuntur $\text{ſin.}\ (30 + z) = \text{coſ.}\ z - \text{ſin.}\ (30 - z)$ & $\text{coſ.}\ (30 + z) = \text{coſ.}\ (30 - z) - \text{ſin.}\ z$, unde Sinus & Coſinus angulorum a 30° ad 60°, hincque omnes majores definiuntur. CAP. VIII.

137. In Tangentibus & Cotangentibus ſimile ſubſidium uſu venit. Cum enim ſit $\text{tang.}\ (a + b) = \frac{\text{tang.}\ a + \text{tang.}\ b}{1 - \text{tang.}\ a.\ \text{tang.}\ b}$, erit $\text{tang.}\ 2a = \frac{2\ \text{tang.}\ a}{1 - (\text{tang.}\ a)^2}$, & $\text{cot.}\ 2a = \frac{\text{cot.}\ a - \text{tang.}\ a}{2}$ unde ex Tangentibus & Cotangentibus Arcuum 30° minorum inveniuntur Cotangentes uſque ad 60°.

Sit jam $a = 30 - b$ erit $2a = 60 - 2b$ & $\text{cot.}\ 2a = \text{tang.}\ (30 + 2b)$; erit ergo $\text{tang.}\ (30 + 2b) = \frac{\text{cot.}\ (30 - b) - \text{tang.}\ (30 - b)}{2}$, unde etiam Tangentes Arcuum 30° majorum obtinentur.

Secantes autem & Coſecantes ex Tangentibus per ſolam ſubtractionem inveniuntur; eſt enim $\text{coſec.}\ z = \text{cot.}\ \frac{1}{2} z - \text{cot.}\ z$, & hinc $\text{ſec.}\ z = \text{cot.}\ (45^o - \frac{1}{2} z) - \text{tang.}\ z$. Ex his ergo luculenter perſpicitur, quomodo canones Sinuum conſtrui potuerint.

138. Ponatur denuo in formulis §. 133, Arcus z infinite parvus, & ſit n numerus infinite magnus i, ut iz obtineat valorem finitum v. Erit ergo $nz = v$; & $z = \frac{v}{i}$, unde $\text{ſin.}\ z = \frac{v}{i}$ & $\text{coſ.}\ z = 1$; his ſubſtitutis fit $\text{coſ.}\ v =$

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LIB. I.

$$\frac{(1+\frac{v\sqrt{-1}}{i})^i + (1-\frac{v\sqrt{-1}}{i})^i}{2}$$; atque $fin.\, v =$

$$\frac{(1+\frac{v\sqrt{-1}}{i})^i - (1-\frac{v\sqrt{-1}}{i})^i}{2\sqrt{-1}}.$$ In Capite autem præcedente vidimus eſſe $(1+\frac{z}{i})^i = e^z$, denotante e baſin Logarithmorum hyperbolicorum: ſcripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit $coſ.\, v =$
$\frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$ & $fin.\, v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}$.
Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Coſinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} = coſ.\, v + \sqrt{-1}.\, fin.\, v$ & $e^{-v\sqrt{-1}} = coſ.\, v - \sqrt{-1}.\, fin.\, v$.

139. Sit jam in iiſdem formulis §. 133, n numerus infinite parvus, ſeu $n = \frac{1}{i}$, exiſtente i numero infinite magno, erit $coſ.\, nz = coſ.\, \frac{z}{i} = 1$ & $fin.\, nz = fin.\, \frac{z}{i} = \frac{z}{i}$; Arcus enim evaneſcentis $\frac{z}{i}$ Sinus eſt ipſi æqualis, Coſinus vero $= 1$. His poſitis habebitur

$$1 = \frac{(coſ.\, z + \sqrt{-1}.\, fin.\, z)^{\frac{1}{i}} + (coſ.\, z - \sqrt{-1}.\, fin.\, z)^{\frac{1}{i}}}{2} \quad \&$$

$$\frac{z}{i} = \frac{(coſ.\, z + \sqrt{-1}.\, fin.\, z)^{\frac{1}{i}} - (coſ.\, z - \sqrt{-1}.\, fin.\, z)^{\frac{1}{i}}}{2\sqrt{-1}}.$$ Sumendis autem Logarithmis hyperbolicis ſupra (125) oſtendimus eſſe $l(1+x) = i(1+x)^{\frac{1}{i}} - i$, ſeu $y^{\frac{1}{i}} = 1 + \frac{1}{i} ly$,

poſito

poſito y loco $1 + x$. Nunc igitur, poſito loco y, partim $cof.z + \sqrt{-1}.\,fin.\,z$ partim $cof.\,z - \sqrt{-1}.\,fin.\,z$, prodibit $1 =$ C A P. VIII.

$$\frac{1+\frac{1}{i}l(cof.z+\sqrt{-1}.fin.z)+1+\frac{1}{i}l(cof.z-\sqrt{-1}.fin.z)}{2}$$

$=1$, ob Logarithmos evaneſcentes, ita ut hinc nil ſequatur. Altera vero æquatio pro Sinu ſuppeditat:

$$\frac{z}{i}=\frac{\frac{1}{i}l(cof.z+\sqrt{-1}.fin.z)-\frac{1}{i}l(cof.z-\sqrt{-1}.fin.z)}{2\sqrt{-1}}$$

ideoque $z = \frac{1}{2\sqrt{-1}}\, l\frac{cof.z+\sqrt{-1}.\,fin.z}{cof.z-\sqrt{-1}.\,fin.z}$, unde patet quemadmodum Logarithmi imaginarii ad Arcus circulares revocentur.

140. Cum ſit $\frac{fin.z}{cof.z} = tang.z$, Arcus z per ſuam Tangentem ita exprimetur ut ſit $z = \frac{1}{2\sqrt{-1}}\, l\frac{1+\sqrt{-1}.\,tang.z}{1-\sqrt{-1}.\,tang.z}$. Supra vero (§. 123) vidimus eſſe $l\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \&c.$. Poſito ergo $x = \sqrt{-1}.\ tang.z$, fiet $z = \frac{tang.z}{1} - \frac{(tang.z)^3}{3} + \frac{(tang.z)^5}{5} - \frac{(tang.z)^7}{7} + \&c.$. Si ergo ponamus $tang.z = t$, ut ſit z Arcus, cujus Tangens eſt t, quem ita indicabimus $A.\ tang.\ t$, ideoque erit $z = A.\ tang.\ t$. Cognita ergo Tangente t erit Arcus reſpondens $z = \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \&c.$. Cum igitur, ſi Tangens t æquetur Radio 1, fiat Arcus $z =$ Arcui 45°, ſeu $z = \frac{\pi}{4}$, erit $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.$, quæ eſt Series a Leibnitzio primum producta, ad valorem Peripheriæ Circuli exprimendum.

141. Quo autem ex hujuſmodi Serie longitudo Arcus Cir-

culi expedite definiri poſſit, perſpicuum eſt pro Tangente t fractionem ſatis parvam ſubſtitui debere. Sic ope hujus Seriei facile reperietur longitudo Arcus z, cujus Tangens t æquetur $\frac{1}{10}$, foret enim iſte Arcus $z = \frac{1}{10} - \frac{1}{3000} + \frac{1}{500000} -$ &c., cujus Seriei valor per approximationem non difficulter in fractione decimali exhiberetur. At vero ex tali Arcu cognito nihil pro longitudine totius Circumferentiæ concludere licebit, cum ratio, quam Arcus, cujus Tangens eſt $= \frac{1}{10}$, ad totam Peripheriam tenet, non ſit aſſignabilis. Hanc ob rem ad Peripheriam indagandam, ejuſmodi Arcus quæri debet, qui ſit ſimul pars aliquota Peripheriæ, & cujus Tangens ſatis exigua commode exprimi queat. Ad hoc ergo inſtitutum ſumi ſolet Arcus 30° cujus Tangens eſt $= \frac{1}{\sqrt{3}}$, quia minorum Arcuum cum Peripheria commenſurabilium Tangentes nimis fiunt irrationales. Quare, ob Arcum $30^o = \frac{\pi}{6}$, erit $\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3\sqrt{3}} + \frac{1}{5 \cdot 3^2\sqrt{3}} -$ &c., & $\pi = \frac{2\sqrt{3}}{1} - \frac{2\sqrt{3}}{3 \cdot 3} + \frac{2\sqrt{3}}{5 \cdot 3^2} - \frac{2\sqrt{3}}{7 \cdot 3^3} +$ &c., cujus Seriei ope valor ipſius π ante exhibitus incredibili labore fuit determinatus.

142. Hic autem labor eo major eſt, quod primum ſinguli termini ſint irrationales, tum vero quiſque tantum, circiter, triplo ſit minor quam præcedens. Huic itaque incommodo ita occurri poterit: ſumatur Arcus 45° ſeu $\frac{\pi}{4}$ cujus valor, etſi per Seriem vix convergentem $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ &c., exprimitur, tamen is retineatur, atque in duos Arcus a & b diſpertiatur ut ſit $a + b = \frac{\pi}{4} = 45^o$. Cum igitur ſit $tang. (a + b) = 1 = \frac{tang.\, a + tang.\, b}{1 - tang.\, a.\ tang.\, b}$ erit $1 - tang.\, a.\, tang.\, b =$

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tang. a + *tang. b* & *tang. b* $= \frac{1 - tang.\,a}{1 + tang.\,a}$. Sit nunc *tang. a* $=$
$\frac{1}{2}$, erit *tang. b* $= \frac{1}{3}$, hinc uterque Arcus *a* & *b* per Seriem rationalem multo magis, quam ſuperior, convergentem exprimetur, eorumque ſumma dabit valorem Arcus $\frac{\pi}{4}$; hinc itaque erit

$$\pi = 4 \left\{ \begin{array}{l} \frac{1}{1.2} - \frac{1}{3.2^3} + \frac{1}{5.2^5} - \frac{1}{7.2^7} + \frac{1}{9.2^9} - \&c. \\ \frac{1}{1.3} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \frac{1}{7.3^7} + \frac{1}{9.3^9} - \&c. \end{array} \right\}$$

hoc ergo modo multo expeditius longitudo ſemicircumferentiæ π inveniri potuiſſet, quam quidem factum eſt ope Seriei ante commemoratæ.

CAPUT IX.

De inveſtigatione Factorum trinomialium.

143. Quemadmodum Factores ſimplices cujuſque Functionis integræ inveniri oporteat, ſupra quidem oſtendimus hoc fieri per reſolutionem æquationum. Si enim propoſita ſit Functio quæcunque integra $\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$, hujuſque quærantur Factores ſimplices formæ $p - qz$, manifeſtum eſt, ſi $p - qz$ fuerit Factor Functionis $\alpha + \beta z + \gamma z^2 + \&c.$, tum, poſito $z = \frac{p}{q}$, quo caſu Factor $p - qz$ fit $= 0$, etiam ipſam Functionem propoſitam evaneſcere debere. Hinc ſi $p - qz$ eſt Factor vel diviſor Functionis $\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$, ſequitur fore hanc

 expreſſionem $\alpha + \frac{\beta p}{q} + \frac{\gamma p^2}{q^2} + \frac{\delta p^3}{q^3} + \frac{\epsilon p^4}{q^4} +$ &c. $= 0$. Unde viciſſim, ſi omnes radices $\frac{p}{q}$ hujus æquationis eruantur, ſingulæ dabunt totidem Factores ſimplices Functionis integræ propoſitæ $\alpha + \beta z + \gamma z^2 + \delta z^3 +$ &c., nempe $p - qz$. Patet autem ſimul numerum Factorum hujuſmodi ſimplicium ex maxima Poteſtate ipſius z definiri.

144. Hoc autem modo plerumque difficulter Factores imaginarii eruuntur, quamobrem hoc Capite methodum peculiarem tradam, cujus ope ſæpenumero Factores ſimplices imaginarii inveniri queant. Quoniam vero Factores ſimplices imaginarii ita ſunt comparati, ut binorum productum fiat reale, hos ipſos Factores imaginarios reperiemus, ſi Factores inveſtigemus duplices, ſeu hujus formæ $p - qz + rzz$, reales quidem, ſed quorum Factores ſimplices ſint imaginarii. Quod ſi enim Functionis $\alpha + \beta z + \gamma z^2 + \delta z^3 +$ &c., conſtent omnes Factores reales duplices hujus formæ trinomialis $p - qz + rzz$, ſimul omnes Factores imaginarii habebuntur.

145. Trinomium autem $p - qz + rzz$ Factores ſimplices habebit imaginarios, ſi fuerit $4pr > qq$ ſeu $\frac{q}{2\sqrt{pr}} < 1$. Cum igitur Sinus & Coſinus Angulorum ſint unitate minores, formula $p - qz + rzz$ Factores ſimplices habebit imaginarios ſi fuerit $\frac{q}{2\sqrt{pr}} =$ Sinui vel Coſinui cujuſpiam Anguli. Sit ergo $\frac{q}{2\sqrt{pr}} = coſ. A\,\varphi$, ſeu $q = 2\sqrt{pr}.\ coſ.\ \varphi$, atque trinomium $p - qz + rzz$ continebit Factores ſimplices imaginarios. Ne autem irrationalitas moleſtiam faceſſat, aſſumo hanc formam $pp - 2pqz.\ coſ.\ \varphi + qqzz$, cujus Factores ſimplices imaginarii erunt hi, $qz - p\,(coſ.\ \varphi + \sqrt{-1}.\ ſin.\ \varphi)$ & $qz - p\,(coſ.\ \varphi - \sqrt{-1}.\ ſin.\ \varphi)$. Ubi quidem patet ſi fuerit $coſ.\ \varphi = \pm 1$, tum ambos Factores, ob $ſin.\ \varphi = 0$, fieri æquales & reales.

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146. Propoſita ergo Functione integra $\alpha + \beta z + \gamma z^2 + \delta z^3 +$ &c., ejus Factores ſimplices imaginarii eruentur, ſi determinentur litteræ p & q cum Angulo φ, ut hoc trinomium $pp - 2pqz.\ cof.\ \varphi + qqzz$ fiat Factor Functionis. Tum enim ſimul inerunt illi Factores ſimplices imaginarii $qz - p(cof.\varphi + \sqrt{-1}.\ fin.\varphi)$ & $qz - p(cof.\varphi - \sqrt{-1}.\ fin.\varphi)$. Quam ob rem Functio propoſita evaneſcet, ſi ponatur tam $z = \frac{p}{q} \times (cof.\varphi + \sqrt{-1}.\ fin.\varphi)$ quam $z = \frac{p}{q}(cof.\varphi - \sqrt{-1}.\ fin.\varphi)$. Hinc, facta ſubſtitutione utraque, duplex naſcetur æquatio, ex quibus tam fractio $\frac{p}{q}$ quam Arcus φ definiri poterunt:

147. Hæ autem ſubſtitutiones loco z faciendæ, etiamſi primo intuitu difficiles videantur, tamen per ea, quæ in Capite præcedente ſunt tradita, ſatis expedite abſolventur. Cum enim fuerit oſtenſum eſſe $(cof.\varphi \pm \sqrt{-1}.fin.\varphi)^n = cof.n\varphi \pm \sqrt{-1} \times fin.n\varphi$, ſequentes formulæ loco ſingularum ipſius z Poteſtatum habebuntur ſubſtituendæ:

pro priori Factore	pro altero Factore
$z = \frac{p}{q}(cof.\ \varphi + \sqrt{-1}.fin.\ \varphi)$	$z = \frac{p}{q}(cof.\ \varphi - \sqrt{-1}.fin.\ \varphi)$
$z^2 = \frac{p^2}{q^2}(cof.2\varphi + \sqrt{-1}.fin.2\varphi)$	$z^2 = \frac{p^2}{q^2}(cof.2\varphi - \sqrt{-1}.fin.2\varphi)$
$z^3 = \frac{p^3}{q^3}(cof.3\varphi + \sqrt{-1}.fin.3\varphi)$	$z^3 = \frac{p^3}{q^3}(cof.3\varphi - \sqrt{-1}.fin.3\varphi)$
$z^4 = \frac{p^4}{q^4}(cof.4\varphi + \sqrt{-1}.fin.4\varphi)$	$z^4 = \frac{p^4}{q^4}(cof.4\varphi - \sqrt{-1}.fin.4\varphi)$
&c.	&c.

Ponatur brevitatis gratia $\frac{p}{q} = r$, factaque ſubſtitutione ſequentes duæ naſcentur æquationes:

$$0 = \begin{cases} \alpha + \beta r.\ cof.\ \varphi + \gamma r^2.\ cof.\ 2\varphi + \delta r^3.\ cof.\ 3\varphi + \&c. \\ + \beta r\sqrt{-1}.fin.\varphi + \gamma r^2\sqrt{-1}.fin.2\varphi + \delta r^3\sqrt{-1}.fin.3\varphi + \&c. \end{cases}$$

$$0 = \begin{cases} \alpha + \beta r.\ cof.\ \varphi + \gamma r^2.\ cof.\ 2\varphi + \delta r^3.\ cof.\ 3\varphi + \&c. \\ - \beta r\sqrt{-1}.fin.\varphi - \gamma r^2\sqrt{-1}.fin.2\varphi - \delta r^3\sqrt{-1}.fin.3\varphi - \&c. \end{cases}$$

LIB. I. 148. Quod ſi hæ duæ æquationes invicem addantur & ſubtrahantur, & poſteriori caſu per $2\sqrt{-1}$ dividantur, prodibunt hæ duæ æquationes reales:

$$0 = \alpha + \beta r.\,cof.\,\varphi + \gamma r^2.\,cof.\,2\varphi + \delta r^3.\,cof.\,3\varphi + \&c.$$
$$0 = \quad \beta r.\,fin.\,\varphi + \gamma r^2.\,fin.\,2\varphi + \delta r^3.\,fin.\,3\varphi + \&c.$$

quæ ſtatim ex forma Functionis propoſitæ

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$$

formari poſſunt, ponendo primum pro unaquaque ipſius z poteſtate $z^n = r^n.\,cof.\,n\varphi$, deinceps $z^n = r^n.\,fin.\,n\varphi$. Sic enim ob $fin.\,0\varphi = 0$ & $cof.\,0\varphi = 1$, pro z^0 ſeu 1 in termino conſtanti priori caſu ponitur 1, poſteriori autem 0. Si ergo ex his duabus æquationibus definiantur incognitæ r & φ, ob $r = \frac{p}{q}$, habebitur Factor Functionis trinomialis $pp - 2pqz.\,cof.\,\varphi + qqzz$, duos Factores ſimplices imaginarios involvens.

149. Si æquatio prior multiplicetur per $fin.\,m\varphi$, poſterior per $cof.\,m\varphi$, atque producta vel addantur vel ſubtrahantur, prodibunt iſtæ duæ æquationes:

$$0 = \alpha.\,fin.\,m\varphi + \beta r.\,fin.\,(m+1)\varphi + \gamma r^2.\,fin.\,(m+2)\varphi + \delta r^3.\,fin.\,(m+3)\varphi + \&c.$$
$$0 = \alpha.\,fin.\,m\varphi + \beta r.\,fin.\,(m-1)\varphi + \gamma r^2.\,fin.\,(m-2)\varphi + \delta r^3.\,fin.\,(m-3)\varphi + \&c.$$

Sin autem æquatio prior multiplicetur per $cof.\,m\varphi$ & poſterior per $fin.\,m\varphi$, per additionem ac ſubtractionem ſequentes emergent æquationes.

$$0 = \alpha.\,cof.\,m\varphi + \beta r.\,cof.\,(m-1)\varphi + \gamma r^2.\,cof.\,(m-2)\varphi + \delta r^3.\,cof.\,(m-3)\varphi + \&c.$$
$$0 = \alpha.\,cof.\,m\varphi + \beta r.\,cof.\,(m+1)\varphi + \gamma r^2.\,cof.\,(m+2)\varphi + \delta r^3.\,cof.\,(m+3)\varphi + \&c.$$

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Hujuſmodi ergo duæ æquationes quæcunque conjunctæ determinabunt incognitas r & φ; quod cum plerumque pluribus modis fieri poſſit, ſimul plures Factores trinomiales obtinentur, iique adeo omnes, quos Functio propoſita in ſe complectitur.

150. Quo uſus harum regularum clarius appareat, quarumdam Functionum ſæpius occurrentium Factores trinomiales hic indagabimus, ut eos, quoties occaſio poſtulaverit, hinc depromere liceat. Sit itaque propoſita hæc Functio $a^n + z^n$, cujus Factores trinomiales formæ $pp - 2pqz. cof. \varphi + qqzz$ determinari oporteat; poſito ergo $r = \frac{p}{q}$, habebuntur hæ duæ æquationes: $0 = a^n + r^n. cof. n\varphi$ & $0 = r^n. fin. n\varphi$, quarum poſterior dat $fin. n\varphi = 0$; unde erit $n\varphi$ Arcus vel hujus formæ $(2k+1)\pi$ vel $2k\pi$, denotante k numerum integrum. Caſus hos ideo diſtinguo, quod eorum Coſinus ſint differentes; priori enim caſu erit $cof.(2k+1)\pi = -1$ poſteriori caſu autem $cof. 2k\pi = +1$. Patet autem priorem formam $n\varphi = (2k+1)\pi$ ſumi debere, quippe quæ dat $cof. n\varphi = -1$; unde fit $0 = a^n - r^n$, hincque porro $r = a = \frac{p}{q}$. Erit ergo $p = a$, $q = 1$, & $\varphi = \frac{(2k+1)\pi}{n}$, unde Functionis $a^n + z^n$ Factor erit $aa - 2az. cof. \frac{(2k+1)\pi}{n} + zz$. Cum igitur pro k numerum quemque integrum ponere liceat, prodeunt hoc modo plures Factores, neque tamen infiniti, quoniam ſi $2k+1$, ultra n augetur, Factores priores recurrunt, quod ex exemplis clarius patebit, cum ſit $cof.(2\pi \pm \varphi) = cof. \varphi$. Deinde ſi n eſt numerus impar, poſito $2k+1 = n$, erit Factor quadratus $aa + 2az + zz$ neque vero hinc ſequitur quadratum $(a+z)^2$ eſſe Factorem Functionis $a^n + z^n$, quoniam (in §. 148) unica æquatio reſultat, qua tantum patet $a + z$ eſſe Diviſorem

 formulæ $a^n + z^n$; quæ regula ſemper eſt tenenda quoties *coſ.* φ fit vel + 1 vel — 1.

Exemplum.

Evolvamus aliquot caſus, quo iſti Factores clarius ob oculos ponatur, atque hos caſus in duas claſſes diſtribuamus, prout n fuerit numerus vel par vel impar.

Si $n = 1$ Formulæ $a + z$ Factor eſt $a + z$	Si $n = 2$ Formulæ $a^2 + z^2$ Factor eſt $a^2 + z^2$
Si $n = 3$ Formulæ $a^3 + z^3$ Factores ſunt $aa - 2az.\ coſ.\ \frac{1}{3}\pi + zz$ $a + z$	Si $n = 4$ Formulæ $a^4 + z^4$ Factores ſunt $aa - 2az.\ coſ.\ \frac{1}{4}\pi + zz$ $aa - 2az.\ coſ.\ \frac{3}{4}\pi + zz$
Si $n = 5$ Formulæ $a^5 + z^5$ Factores ſunt $aa - 2az.\ coſ.\ \frac{1}{5}\pi + zz$ $aa - 2az.\ coſ.\ \frac{3}{5}\pi + zz$ $a + z$	Si $n = 6$ Formulæ $a^6 + z^6$ Factores ſunt $aa - 2az.\ coſ.\ \frac{1}{6}\pi + zz$ $aa - 2az.\ coſ.\ \frac{3}{6}\pi + zz$ $aa - 2az.\ coſ.\ \frac{5}{6}\pi + zz$

Ex quibus exemplis patet omnes Factores obtineri, ſi loco $2k + 1$ omnes numeri impares non majores, quam Exponens

n,

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n, ſubſtituantur, iis vero caſibus quibus Factor quadratus prodit, tantum ejus radicem Factoribus annumerari debere.

151. Si propoſita ſit hæc Functio $a^n - z^n$, ejus Factor trinomialis erit $pp - 2pqz.\ coſ.\varphi + qqzz$, ſi poſito $r = \frac{p}{q}$, fuerit $o = a^n - r^n.\ coſ.n\varphi$ & $o = r^n.\ ſin.n\varphi$. Erit ergo iterum $ſin.\ n\varphi = o$, ideoque $n\varphi = (2k+1)\pi$ vel $n\varphi = 2k\pi$. Hoc autem caſu valor poſterior ſumi debet, ut ſit $coſ.n\varphi = +1$, qui dat $o = a^n - r^n$ & $r = \frac{p}{q} = a$. Habebitur itaque $p = a$, $q = 1$ & $\varphi = \frac{2k\pi}{n}$; unde Factor trinomialis formulæ propoſitæ erit $= aa - 2az.\ coſ.\frac{2k}{n}\pi + zz$; quæ forma, ſi loco $2k$ omnes numeri pares non majores quam n ponantur, ſimul dabit omnes Factores; ubi de Factoribus quadratis idem eſt tenendum quod ante monuimus. Ac primo quidem, poſito $k = o$, prodit Factor $aa - 2az + zz$, pro quo vero radix $a - z$ capi debet. Similiter, ſi n fuerit numerus par & ponatur $2k = n$, prodit $aa + 2az + zz$, unde patet $a + z$ eſſe diviſorem formæ $a^n - z^n$.

Exemplum.

Caſus Exponentis n ut ante tractati ita ſe habebunt, prout n fuerit numerus vel impar vel par.

<table>
<tr><td>Si $n = 1$
Formulæ
$a - z$
ipſa erit Factor
$a - z$</td><td>Si $n = 2$
Formulæ
$a^2 - z^2$
Factores erunt
$a - z$
$a + z$</td></tr>
<tr><td>Si $n = 3$
Formulæ
$a^3 - z^3$
Factores erunt
$a - z$
$aa - 2az.\ coſ.\ \frac{2}{3}\pi + zz$</td><td>Si $n = 4$
Formulæ
$a^4 - z^4$
Factores erunt
$a - z$
$aa - 2az.\ coſ.\ \frac{2}{4}\pi + zz$
$a + z$</td></tr>
<tr><td>Si $n = 5$
Formulæ
$a^5 - z^5$
Factores erunt
$a - z$
$aa - 2az.\ coſ.\ \frac{2}{5}\pi + zz$
$aa - 2az.\ coſ.\ \frac{4}{5}\pi + zz$</td><td>Si $n = 6$
Formulæ
$a^6 - z^6$
Factores erunt
$a - z$
$aa - 2az.\ coſ.\ \frac{2}{6}\pi + zz$
$aa - 2az.\ coſ.\ \frac{4}{6}\pi + zz$
$a + z$</td></tr>
</table>

152. His igitur confirmatur id, quod ſupra jam innuimus, omnem Functionem integram, ſi non in Factores ſimplices reales, tamen in Factores duplices reales reſolvi poſſe. Vidimus enim hanc Functionem indefinitæ dimenſionis $a^n \pm z^n$ ſemper in Factores duplices reales, præter ſimplices reales, reſolvi poſſe. Progrediamur ergo ad Functiones magis compoſitas, uti: $\alpha + \beta z^n + \gamma z^{2n}$, cujus quidem, ſi duos habeat Factores formæ $\eta + \theta z^n$, reſolutio ex præcedentibus abunde patet. Hoc ergo tantum erit efficiendum, ut formæ $\alpha + \beta z^n + \gamma z^{2n}$, eo caſu, quo non habet duos Factores reales formæ $\eta + \theta z^n$,

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reſolutionem in Factores reales, vel ſimplices vel duplices, doceamus. CAP. IX.

153. Conſideremus ergo hanc Functionem: $a^{2n} - 2a^n z^n \times \cos g + z^{2n}$, quæ in duos Factores formæ $n - \theta z^n$ reales reſolvi nequit. Quod ſi ergo ponamus hujus Functionis Factorem duplicem realem eſſe $pp - 2pqz.\ \cos\varphi + qqzz$, poſito $r = \frac{p}{q}$, duæ ſequentes æquationes erunt reſolvendæ: $0 = a^{2n} - 2a^n r^n.\ \cos g.\ \cos n\varphi + r^{2n}.\ \cos 2n\varphi$ & $0 = -2a^n r^n.\ \cos g.\ \sin n\varphi + r^{2n}.\ \sin 2n\varphi$. Vel, loco prioris æquationis ſumatur, ex §. 149, (ponendo $m = 2n$) hæc $0 = a^{2n}.\ \sin 2n\varphi - 2a^n r^n.\ \cos g.\ \sin n\varphi$, quæ cum poſteriori collata dat $r = a$; tum vero erit $\sin 2n\varphi = 2\cos g.\ \sin n\varphi$. At eſt $\sin 2n\varphi = 2\sin n\varphi.\ \cos n\varphi$, unde fit $\cos n\varphi = \cos g$. At eſt ſemper $\cos(2k\pi \pm g) = \cos g$, ex quo habetur $n\varphi = 2k\pi \pm g$ & $\varphi = \frac{2k\pi \pm g}{n}$. Hinc ergo Factor generalis duplex formæ propoſitæ erit $= aa - 2az.\ \cos \frac{2k\pi \pm g}{n} + zz$; atque omnes Factores prodibunt, ſi pro $2k$ omnes numeri pares non majores quam n ſucceſſive ſubſtituantur, uti ex applicatione ad caſus videre licebit.

E X E M P L U M.

Conſideremus ergo caſus quibus n eſt 1, 2, 3, 4, &c., ut ratio Factorum appareat. Erunt ergo

Formulæ

$a^2 - 2az.\ \cos g + zz$

Unicus Factor

$aa - 2az.\ \cos g + zz$

Formulæ

$a^4 - 2a^2z^2.\ \cos g + z^4$

Factores duo

$$aa - 2az.\ \text{coſ.}\ \frac{g}{2} + z^2$$
$$aa - 2az.\ \text{coſ.}\ \frac{2\pi \pm g}{2} + zz \text{ ſeu } aa + 2az.\ \text{coſ.}\ \frac{g}{2} + zz$$

Formulæ

$$a^6 - 2a^3z^3.\ \text{coſ.}\ g + z^6$$

Factores tres

$$aa - 2az.\ \text{coſ.}\ \frac{g}{3} + z^2$$
$$aa - 2az.\ \text{coſ.}\ \frac{2\pi - g}{3} + z^2$$
$$aa - 2az.\ \text{coſ.}\ \frac{2\pi + g}{3} + z^2$$

Formulæ

$$a^8 - 2a^4z^4.\ \text{coſ.}\ g + z^8$$

Factores quatuor

$$aa - 2az.\ \text{coſ.}\ \frac{g}{4} + zz$$
$$aa - 2az.\ \text{coſ.}\ \frac{2\pi - g}{4} + zz$$
$$aa - 2az.\ \text{coſ.}\ \frac{2\pi + g}{4} + zz$$
$$aa - 2az.\ \text{coſ.}\ \frac{4\pi \pm g}{4} + zz \text{ ſeu } aa + 2az.\ \text{coſ.}\ \frac{g}{4} + zz$$

Formulæ

$$a^{10} - 2a^5z^5.\ \text{coſ.}\ g + z^{10}$$

Factores quinque

$$aa - 2az.\ \text{coſ.}\ \frac{g}{5} + zz$$
$$aa - 2az.\ \text{coſ.}\ \frac{2\pi - g}{5} + zz$$
$$aa - 2az.\ \text{coſ.}\ \frac{2\pi + g}{5} + zz$$
$$aa - 2az.\ \text{coſ.}\ \frac{4\pi - g}{5} + zz$$
$$aa - 2az.\ \text{coſ.}\ \frac{4\pi + g}{5} + zz$$

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Confirmatur ergo etiam his exemplis omnem Functionem integram in Factores reales, ſive ſimplices ſive duplices, reſolvi poſſe. CAP. IX.

154. Hinc ulterius progredi licebit ad Functionem hanc $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n}$, quæ certo habebit unum Factorem realem formæ $\eta + \theta z^n$, cujus igitur Factores reales, vel ſimplices vel duplices, exhiberi poſſunt; alter vero multiplicator formæ $\iota + \kappa z^n + \lambda z^{2n}$, utcumque fuerit comparatus, per §. præced. pari modo in Factores reſolvi poterit. Deinde hæc Functio $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \epsilon z^{4n}$, cum perpetuo habeat duos Factores reales formæ hujus $\eta + \theta z^n + \iota z^{2n}$, ſimiliter in Factores, vel ſimplices vel duplices, reales reſolvitur. Quin etiam progredi licet ad formam $\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \epsilon z^{4n} + \zeta z^{5n}$, quæ cum certo habeat unum Factorem formæ $\eta + \theta z^n$, alter Factor erit formæ præcedentis; unde etiam hæc Functio reſolutionem in Factores reales, vel ſimplices vel duplices, admittet. Quare ſi ullum dubium manſiſſet circa hujuſmodi reſolutionem omnium Functionum integrarum, hoc nunc fere penitus tolletur.

155. Traduci vero etiam poteſt hæc in Factores reſolutio ad Series infinitas; ſcilicet, quia vidimus ſupra eſſe $1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c. = e^x$; at vero eſſe $e^x = (1 + \frac{x}{i})^i$, denotante i numerum infinitum, perſpicuum eſt Seriem $1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$ habere Factores infinitos ſimplices inter ſe æquales nempe $1 + \frac{x}{i}$. At ſi ab eadem Serie primus terminus dematur, erit $\frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} +$

&c. $= e^x - 1 = (1 + \frac{x}{i})^i - 1$, cujus formæ cum §. 151 comparatæ, quo ſit $a = 1 + \frac{x}{i}$, $n = i$ & $z = 1$, Factor quicunque erit $= (1 + \frac{x}{i})^2 - 2(1 + \frac{x}{i})\, cof. \frac{2k}{i}\pi + 1$, unde, ſubſtituendo pro $2k$ omnes numeros pares, ſimul omnes Factores prodibunt. Poſito autem $2k = 0$ prodit Factor quadratus $\frac{xx}{ii}$, pro quo autem tantum ob rationes allegatas radix $\frac{x}{i}$ ſumi debet, erit ergo x Factor expreſſionis $e^x - 1$. quod quidem ſponte patet. Ad reliquos Factores inveniendos notari oportet eſſe, ob Arcum $\frac{2k}{i}\pi$ infinite parvum, $cof. \frac{2k}{i}\pi = 1 - \frac{2kk}{ii}\pi\pi$ (134), terminis ſequentibus, ob i numerum infinitum, in nihilum abeuntibus. Hinc erit Factor quilibet $= \frac{xx}{ii} + \frac{4kk}{ii}\pi\pi + \frac{4kk\pi\pi}{i^3}x$, atque adeo forma $e^x - 1$ erit diviſibilis per $1 + \frac{x}{i} + \frac{xx}{4kk\pi\pi}$. Quare expreſſio $e^x - 1 = x(1 + \frac{x}{1.2} + \frac{x^2}{1.2.3} + \frac{x^3}{1.2.3.4} + \&c.)$, præter Factorem x, habebit hos infinitos $(1 + \frac{x}{i} + \frac{xx}{4\pi\pi})(1 + \frac{x}{i} + \frac{xx}{16\pi\pi})(1 + \frac{x}{i} + \frac{xx}{36\pi\pi})(1 + \frac{x}{i} + \frac{xx}{64\pi\pi})$ &c.

156. Cum autem hi Factores contineant partem infinite parvam $\frac{x}{i}$, quæ, cum in ſingulis inſit, atque per multiplicationem omnium, quorum numerus eſt $\frac{1}{2}i$, producat terminum $\frac{x}{2}$, omitti non poteſt. Ad hoc ergo incommodum vitandum conſideremus hanc expreſſionem $e^x - e^{-x} = (1 + \frac{x}{i})^i - (1 - \frac{x}{i})^i = 2(\frac{x}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \&c.)$

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eſt enim $e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \&c.$; quæ cum CAP.IX. §. 151 comparata dat $n = i$, $a = 1 + \frac{x}{i}$ & $z = 1 - \frac{x}{i}$ unde hujus expreſſionis Factor erit $= aa - 2az.\ cof.\frac{2k}{n}\pi + zz = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii})\ cof.\frac{2k}{i}\pi = \frac{4xx}{ii} + \frac{4kk}{ii}\pi\pi - \frac{4kk\pi\pi xx}{i^4}$, ob $cof.\frac{2k}{i}\pi = 1 - \frac{2kk\pi\pi}{ii}$. Functio ergo $e^x - e^{-x}$ diviſibilis erit per $1 + \frac{xx}{kk\pi\pi} - \frac{xx}{ii}$, ubi autem terminus $\frac{xx}{ii}$ tuto omittitur, quia etſi per i multiplicetur, tamen manet infinite parvus. Præterea vero ut ante, ſi $k = 0$, erit primus Factor $= x$. Quocirca, his Factoribus in ordinem redactis, erit $\frac{e^x - e^{-x}}{2} = x(1 + \frac{xx}{\pi\pi})(1 + \frac{xx}{4\pi\pi})(1 + \frac{xx}{9\pi\pi})(1 + \frac{xx}{16\pi\pi})(1 + \frac{xx}{25\pi\pi})\ \&c. = x(1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2....7} + \&c.)$. Singulis ſcilicet Factoribus per multiplicationem conſtantis ejuſmodi formam dedi, ut per actualem multiplicationem primus terminus x reſultet.

157. Eodem modo cum ſit $\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \&c. = \frac{(1 + \frac{x}{i})^i + (1 - \frac{x}{i})^i}{2}$, hujus expreſſionis cum ſuperiori $a^n + z^n$ comparatio dabit $a = 1 + \frac{x}{i}$, $z = 1 - \frac{x}{i}$ & $n = i$: erit ergo Factor quicunque $= aa - 2az \times cof.\frac{2k+1}{n}\pi + zz = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii})\ cof.\frac{2k+1}{i}\pi$. Eſt

LIB. I. autem *coſ.* $\frac{2k+1}{i}\pi = 1 - \frac{(2k+1)^2\pi\pi}{2ii}$, unde forma Factoris erit $= \frac{4xx}{ii} + \frac{(2k+1)^2\pi^2}{ii}$, evaneſcente termino cujus denominator eſt i^4. Quoniam ergo omnis Factor expreſſionis $1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} +$ &c. hujuſmodi formam habere debet $1 + \alpha xx$, quo Factor inventus ad hanc formam reducatur, dividi debet per $\frac{(2k+1)^2\pi^2}{ii}$: hinc Factor formæ propoſitæ erit $1 + \frac{4xx}{(2k+1)^2\pi\pi}$, ex eoque omnes Factores infiniti invenientur, ſi loco $2k+1$ ſucceſſive omnes numeri impares ſubſtituantur. Hanc ob rem erit

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \&c. =$$
$$\left(1 + \frac{4xx}{\pi\pi}\right)\left(1 + \frac{4xx}{9\pi\pi}\right)\left(1 + \frac{4xx}{25\pi\pi}\right)\left(1 + \frac{4xx}{49\pi\pi}\right) \&c.$$

158. Si x fiat quantitas imaginaria, formulæ hæ exponentiales in Sinum & Coſinum cujuſpiam Arcus realis abeunt. Sit enim $x = z\sqrt{-1}$; erit $\frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}} =$ *ſin.* $z = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.3\ldots7} +$ &c., quæ adeo expreſſio hos habet Factores numero infinitos $z\left(1 - \frac{zz}{\pi\pi}\right)\left(1 - \frac{zz}{4\pi\pi}\right)\left(1 - \frac{zz}{9\pi\pi}\right)\left(1 - \frac{zz}{16\pi\pi}\right)\left(1 - \frac{zz}{25\pi\pi}\right)$ &c., ſeu erit *ſin.* $z = z\left(1 - \frac{z}{\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)\left(1 + \frac{z}{2\pi}\right)\left(1 - \frac{z}{3\pi}\right)\left(1 + \frac{z}{3\pi}\right)$ &c.. Quoties ergo Arcus z ita eſt comparatus, ut quiſpiam Factor evaneſcat, quod fit ſi $z = 0$, $z = \pm\pi$, $z = \pm 2\pi$, & generaliter ſi $z = \pm k\pi$, denotante k numerum quemcunque integrum, ſimul

Sinus

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nus ejus Arcus debet eſſe $= 0$, quod quidem ita patet, ut hinc iſtos Factores a poſteriori eruere licuiſſet. CAP. IX.

Simili modo, cum ſit $\frac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2} = cof. z$, erit quoque $cof. z = (1 - \frac{4zz}{\pi\pi})(1 - \frac{4zz}{9\pi\pi})(1 - \frac{4zz}{25\pi\pi})(1 - \frac{4zz}{49\pi\pi})$ &c., ſeu, his Factoribus in binos reſolvendis, erit quoque $cof. z = (1 - \frac{2z}{\pi})(1 + \frac{2z}{\pi})(1 - \frac{2z}{3\pi})(1 + \frac{2z}{3\pi})(1 - \frac{2z}{5\pi})(1 + \frac{2z}{5\pi})$ &c., ex qua pari modo patet, ſi fuerit $z = \pm \frac{(2k+1)}{2}\pi$, fore $cof. z = 0$, id quod etiam ex natura Circuli liquet.

159. Ex §. 152. etiam inveniri poſſunt Factores hujus expreſſionis $e^x - 2\,cof. g + e^{-x} = 2(1 - cof. g + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} +$ &c.). Tranſit enim hæc expreſſio in hanc $(1 + \frac{x}{i})^i - 2\,cof. g + (1 - \frac{x}{i})^i$, quæ cum illa forma comparata dat $2n = i$, $a = 1 + \frac{x}{i}$ & $z = 1 - \frac{x}{i}$, unde Factor quicunque hujus formulæ erit $= aa - 2az.\ cof. \frac{2k\pi \pm g}{n} + zz = 2 + \frac{2xx}{ii} - 2(1 - \frac{xx}{ii}).\ cof. \frac{2(2k\pi \pm g)}{i}$: at eſt $cof. \frac{2(2k\pi \pm g)}{i} = 1 - \frac{2(2k\pi \pm g)^2}{ii}$, unde Factor erit $= \frac{4xx}{ii} + \frac{4(2k\pi \pm g)^2}{ii}$, ſeu hujus formæ $1 + \frac{xx}{(2k\pi \pm g)^2}$. Si ergo expreſſio per $2(1 - cof. g)$ dividatur, ut in Serie infinita terminus conſtans ſit $= 1$, erit, ſumendis omnibus Factoribus, $\frac{e^x - 2\,cof. g + e^{-x}}{2(1 - cof. g)} = (1 + \frac{xx}{gg})(1 + \frac{xx}{(2\pi - g)^2})$

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 $(1+\frac{xx}{(2\pi+g)^2})(1+\frac{xx}{(4\pi-g)^2})(1+\frac{xx}{(4\pi+g)^2})$ $(1+\frac{xx}{(6\pi-g)^2})(1+\frac{xx}{(6\pi+g)^2})$ &c.. Atque, ſi loco x ponatur $z\surd-1$, erit $\frac{cof.\, z - cof.\, g}{1 - cof.\, g} = (1-\frac{z}{g})(1+\frac{z}{g})$ $(1-\frac{z}{2\pi-g})(1+\frac{z}{2\pi-g})(1-\frac{z}{2\pi+g})(1+\frac{z}{2\pi+g})$ $(1-\frac{z}{4\pi-g})(1+\frac{z}{4\pi-g})$ &c. $= 1 - \frac{zz}{1.2\,(1-cof.g)} + \frac{z^4}{1.2.3.4\,(1-cof.g)} - \frac{z^6}{1.2\ldots 6\,(1-cof.g)} +$ &c.. Hujus adeo Seriei in infinitum continuatæ Factores omnes cognoſcuntur.

160. Commode etiam hujuſmodi Functionis $e^{b+x} \pm e^{c-x}$ Factores inveniri omneſque aſſignari poſſunt. Tranſmutatur enim in hanc formam $(1+\frac{b+x}{i})^i \pm (1+\frac{c-x}{i})^i$, quæ comparata cum forma $a^i \pm z^i$, Factorem habebit $aa - 2az.\, cof.\frac{m\pi}{i} + zz$, denotante m numerum imparem ſi valeat ſignum ſuperius, contra vero numerum parem. Cum autem, ob i numerum infinite magnum, ſit $cof.\frac{m\pi}{i} = 1 - \frac{mm\pi\pi}{2ii}$, erit Factor ille generalis $=(a-z)^2 + \frac{mm\pi\pi}{ii}az$. At hoc caſu erit $a = 1 + \frac{b+x}{i}$ & $z = 1 + \frac{c-x}{i}$, unde fit $(a-z)^2 = \frac{(b-c+2x)^2}{ii}$ & $az = 1 + \frac{b-c}{i} + \frac{bc+(c-b)x-xx}{ii}$; ideoque Factor erit per ii multiplicatus $=(b-c)^2 + 4(b-c)x + 4xx + mm\pi\pi$, neglectis terminis per i vel ii diviſis, quoniam jam omnis generis termini adſunt, præ quibus hi evaneſcerent. Termino ergo conſtante ad unitatem per diviſionem reducto erit Factor $= 1 + \frac{4(b-c)x + 4xx}{mm\pi\pi + (b-c)^2}$.

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161. Nunc, quoniam in omnibus Factoribus terminus conſtans eſt $= 1$, ipſa Functio $e^{b+x} \pm e^{c-x}$ per ejuſmodi conſtantem dividi debet, ut terminus conſtans fiat $= 1$, ſeu ut ejus valor, poſito $x = 0$, fiat $= 1$; talis Diviſor erit $e^b \pm e^c$, & hanc ob rem expreſſio hæc $\frac{e^{b+x} \pm e^{c-x}}{e^b \pm e^c}$ per Factores numero infinitos exponi poterit. Erit ergo, ſi valeat ſignum ſuperius atque m denotet numerum imparem, $\frac{e^{b+x}+e^{c-x}}{e^b+e^c} = \left(1 + \frac{4(b-c)x+4xx}{\pi\pi+(b-c)^2}\right)\left(1 + \frac{4(b-c)x+4xx}{9\pi\pi+(b-c)^2}\right)$ $\left(1 + \frac{4(b-c)x+4xx}{25\pi\pi+(b-c)^2}\right)$ &c.; ſin autem ſignum inferius valeat, atque ideo m denotet numerum parem, caſuque $m = 0$ radix Factoris quadrati ponatur, erit $\frac{e^{b+x}-e^{c-x}}{e^b-e^c} =$ $\left(1 + \frac{2x}{b-c}\right)\left(1 + \frac{4(b-c)x+4xx}{4\pi\pi+(b-c)^2}\right)\left(1 + \frac{4(b-c)x+4xx}{16\pi\pi+(b-c)^2}\right)$ $\left(1 + \frac{4(b-c)x+4xx}{36\pi\pi+(b-c)^2}\right)$ &c. Cap. IX.

162. Ponatur $b = 0$, quod ſine detrimento univerſalitatis fieri poteſt, eritque $\frac{e^x + e^c e^{-x}}{1+e^c} = \left(1 - \frac{4cx+4xx}{\pi\pi+cc}\right)$ $\left(1 - \frac{4cx+4xx}{9\pi\pi+cc}\right)\left(1 - \frac{4cx+4xx}{25\pi\pi+cc}\right)$ &c.; $\frac{e^x - e^c e^{-x}}{1-e^c}$ $= \left(1 - \frac{2x}{c}\right)\left(1 - \frac{4cx+4xx}{4\pi\pi+cc}\right)\left(1 - \frac{4cx+4xx}{16\pi\pi+cc}\right)$ $\left(1 - \frac{4cx+4xx}{36\pi\pi+cc}\right)$ &c.. Jam ponatur c negativum, atque habebuntur hæ duæ æquationes: $\frac{e^x + e^{-c}e^{-x}}{1+e^{-c}} =$

Lib. I. $\left(1+\frac{4cx+4xx}{\pi\pi+cc}\right)\left(1+\frac{4cx+4xx}{9\pi\pi+cc}\right)\left(1+\frac{4cx+4xx}{25\pi\pi+cc}\right)$ &c.,

$$\frac{e^{x}-e^{-c}e^{-x}}{1-e^{-c}}=\left(1+\frac{2x}{c}\right)\left(1+\frac{4cx+4xx}{4\pi\pi+cc}\right)$$

$\left(1+\frac{4cx+4xx}{16\pi\pi+cc}\right)\left(1+\frac{4cx+4xx}{36\pi\pi+cc}\right)$ &c.. Multiplicetur forma prima per tertiam, ac prodibit $\frac{e^{2x}+e^{-2x}+e^{c}+e^{-c}}{2+e^{c}+e^{-c}}$; ponatur vero y loco $2x$, eritque $\frac{e^{y}+e^{-y}+e^{c}+e^{-c}}{2+e^{c}+e^{-c}}=\left(1-\frac{2cy+yy}{\pi\pi+cc}\right)$ $\left(1+\frac{2cy+yy}{\pi\pi+cc}\right)\left(1-\frac{2cy+yy}{9\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{9\pi\pi+cc}\right)\left(1-\frac{2cy+yy}{25\pi\pi+cc}\right)$ $\left(1+\frac{2cy+yy}{25\pi\pi+cc}\right)$ &c.. Multiplicetur prima forma per quartam, erit productum $=\frac{e^{2x}-e^{-2x}+e^{c}-e^{-c}}{e^{c}-e^{-c}}$; ponatur y pro $2x$, eritque $\frac{e^{y}-e^{-y}+e^{c}-e^{-c}}{e^{c}-e^{-c}}=$ $\left(1+\frac{y}{c}\right)\left(1-\frac{2cy+yy}{\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{4\pi\pi+cc}\right)\left(1-\frac{2cy+yy}{9\pi\pi+cc}\right)$ $\left(1+\frac{2cy+yy}{16\pi\pi+cc}\right)\left(1-\frac{2cy+yy}{25\pi\pi+cc}\right)$ &c.. Si ſecunda forma per tertiam multiplicetur, prodibit eadem æquatio niſi quod c capiendum ſit negativum, erit nempe

$$\frac{e^{c}-e^{-c}-e^{y}+e^{-y}}{e^{c}-e^{-c}}=\left(1-\frac{y}{c}\right)\left(1+\frac{2cy+yy}{\pi\pi+cc}\right)$$

$\left(1-\frac{2cy+yy}{4\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{9\pi\pi+cc}\right)\left(1-\frac{2cy+yy}{16\pi\pi+cc}\right)$ $\left(1+\frac{2cy+yy}{25\pi\pi+cc}\right)\left(1-\frac{2cy+yy}{36\pi\pi+cc}\right)$ &c.. Multiplicetur de-

nique forma ſecunda per quartam eritque $\frac{e^y+e^{-y}-e^c-e^{-c}}{2-e^c-e^{-c}}$ Cap.IX.

$=(1-\frac{yy}{cc})(1-\frac{2cy+yy}{4\pi\pi+cc})(1+\frac{2cy+yy}{4\pi\pi+cc})(1-\frac{2cy+yy}{16\pi\pi+cc})$
$(1+\frac{2cy+yy}{16\pi\pi+cc})(1-\frac{2cy+yy}{36\pi\pi+cc})(1+\frac{2cy+yy}{36\pi\pi+cc})$ &c.

163. Hæ quatuor combinationes nunc commode ad Circulum transferri poſſunt, ponendo $c=g\sqrt{-1}$ & $y=v\sqrt{-1}$: erit enim $e^{v\sqrt{-1}}+e^{-v\sqrt{-1}}=2\,cof.\,v$; $e^{v\sqrt{-1}}-e^{-v\sqrt{-1}}=2\sqrt{-1}.\,fin.\,v.$ & $e^{g\sqrt{-1}}+e^{-g\sqrt{-1}}=2\,cof.\,g$; $e^{g\sqrt{-1}}-e^{-g\sqrt{-1}}=2\sqrt{-1}.\,fin.\,g.$ Hinc prima combinatio dabit $\frac{cof.\,v+cof.\,g}{1+cof.\,g}=1-\frac{vv}{1.2(1+cof.\,g)}+\frac{v^4}{1.2.3.4(1+cof.\,g)}-\frac{v^6}{1.2\ldots.6(1+cof.\,g)}+\&c.=(1+\frac{2gv-vv}{\pi\pi-gg})$
$(1-\frac{2gv-vv}{\pi\pi-gg})(1+\frac{2gv-vv}{9\pi\pi-gg})(1-\frac{2gv-vv}{9\pi\pi-gg})$
$(1+\frac{2gv-vv}{25\pi\pi-gg})(1-\frac{2gv-vv}{25\pi\pi-gg})\,\&c.=(1+\frac{v}{\pi-g})$
$(1-\frac{v}{\pi+g})(1-\frac{v}{\pi-g})(1+\frac{v}{\pi+g})(1+\frac{v}{3\pi-g})$
$(1-\frac{v}{3\pi+g})(1-\frac{v}{3\pi-g})(1+\frac{v}{3\pi+g})\,\&c.=$
$(1-\frac{vv}{(\pi-g)^2})(1-\frac{vv}{(\pi+g)^2})(1-\frac{vv}{(3\pi-g)^2})$
$(1-\frac{vv}{(3\pi+g)^2})(1-\frac{vv}{(5\pi-g)^2})\,\&c.$. Quarta vero combinatio dat $\frac{cof.\,v-cof.\,g}{1-cof.\,g}=1-\frac{vv}{1.2(1-cof.\,g)}+\frac{v^4}{1.2.3.4(1-cof.\,g.)}-\frac{v^6}{1.2\ldots.6(1-cof.\,g)}+\&c.=(1-\frac{vv}{gg})$
$(1+\frac{2gv-vv}{4\pi\pi-gg})(1-\frac{2gv-vv}{4\pi\pi-gg})(1+\frac{2gv-vv}{16\pi\pi-gg})$

$(1-\frac{2gv-vv}{16\pi\pi-gg})$ &c. $=(1-\frac{v}{g})(1+\frac{v}{g})(1+\frac{v}{2\pi-g})$
$(1-\frac{v}{2\pi+g})(1-\frac{v}{2\pi-g})(1+\frac{v}{2\pi+g})(1+\frac{v}{4\pi-g})$
$(1-\frac{v}{4\pi+g})$ &c. $=(1-\frac{vv}{gg})(1-\frac{vv}{(2\pi-g)^2})$
$(1-\frac{vv}{(2\pi+g)^2})(1-\frac{vv}{(4\pi-g)^2})(1-\frac{vv}{(4\pi+g)^2})$ &c.

Secunda combinatio dat $\frac{\text{ſin. } g+\text{ſin. } v}{\text{ſin. } g}=1+\frac{v}{\text{ſin.} g}-\frac{v^3}{1.2.3\,\text{ſin.} g}+\frac{v^5}{1.2\ldots 5\,\text{ſin.} g}-$ &c. $=(1+\frac{v}{g})(1+\frac{2gv-vv}{\pi\pi-gg})$
$(1-\frac{2gv-vv}{4\pi\pi-gg})(1+\frac{2gv-vv}{9\pi\pi-gg})(1+\frac{2gv-vv}{16\pi\pi-gg})$ &c.
$=(1+\frac{v}{g})(1+\frac{v}{\pi-g})(1-\frac{v}{\pi+g})(1-\frac{v}{2\pi-g})$
$(1+\frac{v}{2\pi+g})(1+\frac{v}{3\pi-g})(1-\frac{v}{3\pi+g})(1-\frac{v}{5\pi-g})$ &c.

Ac ſumto v negativo prodit tertia combinatio.

164. Ipſæ vero etiam expreſſiones in §. 162 primum inventæ ad Arcus circulares traduci poſſunt hoc modo: cum ſit

$$\frac{e^x+e^c e^{-x}}{1+e^c}=\frac{(1+e^{-c})(e^x+e^c e^{-x})}{2+e^c+e^{-c}}=$$

$$\frac{e^x+e^{-x}+e^{c-x}+e^{-c+x}}{2+e^c+e^{-c}},$$

ſi ponamus $c=g\sqrt{-1}$ & $x=z\sqrt{-1}$, hæc expreſſio abit in hanc $\frac{\text{coſ. } z+\text{coſ. }(g-z)}{1+\text{coſ. } g}=\text{coſ. } z+\frac{\text{ſin. } g.\ \text{ſin. } z}{1+\text{coſ. } g}$. Erit ergo (ob $\frac{\text{ſin. } g}{1+\text{coſ. } g}=\text{tang. }\frac{1}{2}g$) $\text{coſ. } z+\text{tang. }\frac{1}{2}g.\ \text{ſin. } z=1+\frac{z}{1}\text{ tang. }\frac{1}{2}g-\frac{zz}{1.2}-\frac{z^3}{1.2.3}\text{ tang. }\frac{1}{2}g+\frac{z^4}{1.2.3.4}+\frac{z^5}{1.2.3.4.5}\text{ tang. }\frac{1}{2}g-$ &c.
$=(1+\frac{4gz-4zz}{\pi\pi-gg})(1+\frac{4gz-4zz}{9\pi\pi-gg})(1+\frac{4gz-4zz}{25\pi\pi-gg})$ &c.

$= (1 + \frac{2z}{\pi - g})(1 - \frac{2z}{\pi + g})(1 + \frac{2z}{3\pi - g})(1 - \frac{2z}{3\pi + g})$ $(1 + \frac{2z}{5\pi - g})(1 - \frac{2z}{5\pi + g})$ &c.. Simili modo altera expreſſio, ſi Numerator & Denominator per $1 - e^{-c}$ multiplicetur, abit in $\frac{e^x + e^{-x} - e^{c-x} - e^{-c+x}}{2 - e^c - e^{-c}}$; quæ, facto $c = g\sqrt{-1}$ & $x = z\sqrt{-1}$, dat $\frac{cof. z - cof. (g - z)}{1 - cof. g} =$ $cof. z - \frac{ſin. g. ſin. z}{1 - cof. g} = cof. z - \frac{ſin. z}{tang. \frac{1}{2} g}$. Erit ergo $cof. z -$ $cot. \frac{1}{2} g. ſin. z = 1 - \frac{z}{1} cot. \frac{1}{2} g - \frac{zz}{1.2} + \frac{z^3}{1.2.3} cot. \frac{1}{2} g +$ $\frac{z^4}{1.2.3.4} - \frac{z^5}{1.2....5} cot. \frac{1}{2} g + \&c. = (1 - \frac{2z}{g})(1 + \frac{4gz - 4zz}{4\pi\pi - gg})$ $(1 + \frac{4gz - 4zz}{16\pi\pi - gg})(1 + \frac{4gz - 4zz}{36\pi\pi - gg})$ &c. $= (1 - \frac{2z}{g})$ $(1 + \frac{2z}{2\pi - g})(1 - \frac{2z}{2\pi + g})(1 + \frac{2z}{4\pi - g})(1 - \frac{2z}{4\pi + g})$ &c. Quod ſi ergo ponatur $v = 2z$ ſeu $z = \frac{1}{2} v$; habebitur

$\frac{cof. \frac{1}{2}(g - v)}{cof. \frac{1}{2} g} = cof. \frac{1}{2} v + tang. \frac{1}{2} g. ſin. \frac{1}{2} v =$ $(1 + \frac{v}{\pi - g})(1 - \frac{v}{\pi + g})(1 + \frac{v}{3\pi - g})(1 - \frac{v}{3\pi + g})$ &c.;

$\frac{cof. \frac{1}{2}(g + v)}{cof. \frac{1}{2} g} = cof. \frac{1}{2} v - tang. \frac{1}{2} g. ſin. \frac{1}{2} v =$ $(1 - \frac{v}{\pi - g})(1 + \frac{v}{\pi + g})(1 - \frac{v}{3\pi - g})(1 + \frac{v}{3\pi + g})$ &c.;

$\frac{ſin. \frac{1}{2}(g - v)}{ſin. \frac{1}{2} g} = cof. \frac{1}{2} v - cot. \frac{1}{2} g. ſin. \frac{1}{2} v =$ $(1 - \frac{v}{g})(1 + \frac{v}{2\pi - g})(1 - \frac{v}{2\pi + g})(1 + \frac{v}{4\pi - g})$ &c.;

$$\frac{\text{ſin.}\ \frac{1}{2}(g+v)}{\text{ſin.}\ \frac{1}{2}g} = \text{coſ.}\ \tfrac{1}{2}v + \text{cot.}\ \tfrac{1}{2}g.\ \text{ſin.}\ \tfrac{1}{2}v =$$
$$\left(1+\frac{v}{g}\right)\left(1-\frac{v}{2\pi-g}\right)\left(1+\frac{v}{2\pi+g}\right)\left(1-\frac{v}{4\pi-g}\right)\ \&c.$$
Quorum Factorum lex progreſſionis ſatis eſt ſimplex & uniformis ; atque ex his expreſſionibus per multiplicationem oriuntur eæ ipſæ, quæ in §. præcedente ſunt inventæ.

C A P U T X.

De uſu Factorum inventorum in definiendis ſummis Serierum infinitarum.

165. Si fuerit $1 + Az + Bz^2 + Cz^3 + Dz^4 + \&c. = (1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)\ \&c.$, hi Factores, ſive ſint numero finiti ſive infiniti, ſi in ſe actu multiplicentur, illam expreſſionem $1 + Az + Bz^2 + Cz^3 + Dz^4 + \&c.$ producere debent. Æquabitur ergo coëfficiens A ſummæ omnium quantitatum $\alpha + \beta + \gamma + \delta + \epsilon + \&c.$. Coëfficiens vero B æqualis erit ſummæ productorum ex binis, eritque $B = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta + \&c.$. Tum vero coëfficiens C æquabitur ſummæ productorum ex ternis, nempe erit $C = \alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta + \&c.$. Atque ita porro erit $D =$ ſummæ productorum ex quaternis, $E =$ ſummæ productorum ex quinis, &c., id quod ex Algebra communi conſtat.

166. Quia ſumma quantitatum $\alpha + \beta + \gamma + \delta + \&c.$, datur una cum ſumma productorum ex binis, hinc ſumma Quadratorum $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c.$, inveniri poterit, quippe quæ æqualis eſt Quadrato ſummæ demtis duplicibus productis ex binis. Simili modo ſumma Cuborum, Biquadratorum & altiorum Poteſtatum definiri poteſt: ſi enim ponamus

$P =$

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$$P = \alpha + \beta + \gamma + \delta + \epsilon + \&c.$$
$$Q = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 + \&c.$$
$$R = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \epsilon^3 + \&c.$$
$$S = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \epsilon^4 + \&c.$$
$$T = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \epsilon^5 + \&c.$$
$$V = \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \epsilon^6 + \&c.$$
&c.

Valores P, Q, R, S, T, V &c. ſequenti modo ex cognitis A, B, C, D, &c. determinabuntur.

$$P = A$$
$$Q = AP - 2B$$
$$R = AQ - BP + 3C$$
$$S = AR - BQ + CP - 4D$$
$$T = AS - BR + CQ - DP + 5E$$
$$V = AT - BS + CR - DQ + EP - 6F$$
&c.

quarum formularum veritas examine inſtituto facile agnoſcitur.

167. Cum igitur ſupra (§. 156.) invenerimus eſſe: $\frac{e^x - e^{-x}}{2} = x\left(1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2....7} + \&c.\right) = x\left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)\left(1 + \frac{xx}{25\pi\pi}\right)$ &c., erit $1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.3....7} + \&c. = \left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)$ &c. Ponatur $xx = \pi\pi z$, eritque $1 + \frac{\pi\pi}{1.2.3}z + \frac{\pi^4}{1.2.3.4.5}z^2 + \frac{\pi^6}{1.2.3....7}z^3 + \&c. = (1+z)(1+\frac{1}{4}z)(1+\frac{1}{9}z)(1+\frac{1}{16}z)$

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 $(1 + \frac{1}{25} z)$ &c.. Facta ergo applicatione ſuperioris regulæ ad hunc caſum, erit $A = \frac{\pi\pi}{6}$; $B = \frac{\pi^4}{120}$; $C = \frac{\pi^6}{5040}$; $D = \frac{\pi^8}{362880}$ &c.. Quod ſi ergo ponatur

$$P = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \&c.$$
$$Q = 1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \frac{1}{36^2} + \&c.$$
$$R = 1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \frac{1}{36^3} + \&c.$$
$$S = 1 + \frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \frac{1}{25^4} + \frac{1}{36^4} + \&c.$$
$$T = 1 + \frac{1}{4^5} + \frac{1}{9^5} + \frac{1}{16^5} + \frac{1}{25^5} + \frac{1}{36^5} + \&c.$$
&c.

atque harum litterarum valores ex A, B, C, D, &c. determinentur, prodibit.

$$P = \frac{\pi\pi}{6}$$
$$Q = \frac{\pi^4}{90}$$
$$R = \frac{\pi^6}{945}$$
$$S = \frac{\pi^8}{9450}$$
$$T = \frac{\pi^{10}}{93555}$$
&c.

168. Patet ergo omnium Serierum infinitarum in hac formæ generali $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c.$, contentarum, quoties n fuerit numerus par, ſummam ope Peripheriæ Circuli π exhiberi poſſe; habebit enim ſemper ſumma Seriei ad π^n rationem

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rationalem. Quo autem valor harum ſummarum clarius perſpiciatur, plures hujuſmodi Serierum ſummas commodiori modo expreſſas hic adjiciam.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\&c. = \frac{2^0}{1.2.3}\cdot\frac{1}{1}\,\pi^2$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\&c. = \frac{2^2}{1.2.3.4.5}\cdot\frac{1}{3}\,\pi^4$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\&c. = \frac{2^4}{1.2.3\ldots7}\cdot\frac{1}{3}\,\pi^6$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\&c. = \frac{2^6}{1.2.3\ldots9}\cdot\frac{3}{5}\,\pi^8$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\&c. = \frac{2^8}{1.2.3\ldots11}\cdot\frac{5}{3}\,\pi^{10}$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\&c. = \frac{2^{10}}{1.2.3\ldots13}\cdot\frac{691}{105}\,\pi^{12}$$

$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\frac{1}{5^{14}}+\&c. = \frac{2^{12}}{1.2.3\ldots15}\cdot\frac{35}{1}\,\pi^{14}$$

$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\frac{1}{5^{16}}+\&c. = \frac{2^{14}}{1.2.3\ldots17}\cdot\frac{3617}{15}\,\pi^{16}$$

$$1+\frac{1}{2^{18}}+\frac{1}{3^{18}}+\frac{1}{4^{18}}+\frac{1}{5^{18}}+\&c. = \frac{2^{16}}{1.2.3\ldots19}\cdot\frac{43867}{21}\,\pi^{18}$$

$$1+\frac{1}{2^{20}}+\frac{1}{3^{20}}+\frac{1}{4^{20}}+\frac{1}{5^{20}}+\&c. = \frac{2^{18}}{1.2.3\ldots21}\cdot\frac{1222277}{55}\,\pi^{20}$$

$$1+\frac{1}{2^{22}}+\frac{1}{3^{22}}+\frac{1}{4^{22}}+\frac{1}{5^{22}}+\&c. = \frac{2^{20}}{1.2.3\ldots23}\cdot\frac{854513}{3}\,\pi^{22}$$

$$1+\frac{1}{2^{24}}+\frac{1}{3^{24}}+\frac{1}{4^{24}}+\frac{1}{5^{24}}+\&c. = \frac{2^{22}}{1.2.3.\ldots\ldots\ldots25}\cdot\frac{1181820455}{273}\,\pi^{24}$$

$$1+\frac{1}{2^{26}}+\frac{1}{3^{26}}+\frac{1}{4^{26}}+\frac{1}{5^{26}}+\&c. = \frac{2^{24}}{1.2.3.\ldots\ldots\ldots27}\cdot\frac{76977927}{1}\,\pi^{26}.$$

Hucuſque iſtos Poteſtatum ipſius π Exponentes artificio alibi exponendo continuare licuit, quod ideo hic adjunxi, quod

LIB. I. Seriei fractionum primo intuitu perquam irregularis 1, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{3}{5}$, $\frac{5}{3}$, $\frac{691}{105}$, $\frac{35}{1}$, &c. in plurimis occaſionibus eximius eſt uſus.

169. Tractemus eodem modo æquationem §. 157 inventam, ubi erat $\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} +$ &c. $= (1 + \frac{4xx}{\pi\pi})(1 + \frac{4xx}{9\pi\pi})(1 + \frac{4xx}{25\pi\pi})(1 + \frac{4xx}{49\pi\pi})$ &c.. Poſito ergo $xx = \frac{\pi\pi z}{4}$ erit $1 + \frac{\pi\pi}{1.2.4} z + \frac{\pi^4}{1.2.3.4.4^2} zz + \frac{\pi^6}{1.2....6.4^3} z^3 +$ &c. $= (1 + z)(1 + \frac{1}{9} z)(1 + \frac{1}{25} z)(1 + \frac{1}{49} z)$ &c.. Unde, facta applicatione, erit $A = \frac{\pi\pi}{1.2.4}$; $B = \frac{\pi^4}{1.2.3.4.4^2}$; $C = \frac{\pi^6}{1.2.3....6.4^3}$; &c.. Quod ſi ergo ponamus

$$P = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \&c.$$
$$Q = 1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \frac{1}{81^2} + \&c.$$
$$R = 1 + \frac{1}{9^3} + \frac{1}{25^3} + \frac{1}{49^3} + \frac{1}{81^3} + \&c.$$
$$S = 1 + \frac{1}{9^4} + \frac{1}{25^4} + \frac{1}{49^4} + \frac{1}{81^4} + \&c.$$
&c.

reperientur ſequentes pro P, Q, R, S, &c., valores:

$$P = \frac{1}{1} \cdot \frac{\pi^2}{2^3}; \qquad Q = \frac{2}{1.2.3} \cdot \frac{\pi^4}{2^5}$$
$$R = \frac{16}{1.2.3.4.5} \cdot \frac{\pi^6}{2^7}; \qquad S = \frac{272}{1.2.3....7} \cdot \frac{\pi^8}{2^9}$$

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$T = \frac{7936}{1.2.3....9} \cdot \frac{\pi^{10}}{2^{11}}$; $V = \frac{353792}{1.2.3....11} \cdot \frac{\pi^{12}}{2^{13}}$

$W = \frac{22368256}{1.2.3....13} \cdot \frac{\pi^{14}}{2^{15}}$.

170. Eædem ſummæ Poteſtatum numerorum imparium inveniri poſſunt ex ſummis præcedentibus, in quibus omnes numeri occurrunt; ſi enim fuerit $M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} +$ &c., erit ubique, per $\frac{1}{2^n}$ multiplicando, $\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} +$ &c., quæ Series numeros tantum pares continens, ſi a priori ſubtrahatur, relinquet numeros impares, eritque ideo $M - \frac{M}{2^n} = \frac{2^n - 1}{2^n} M = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} +$ &c.. Quod ſi autem Series $\frac{M}{2^n}$ bis ſumta ſubtrahatur ab M ſigna prodibunt alternantia, eritque $M - \frac{2M}{2^n} = \frac{2^{n-1} - 1}{2^{n-1}} M = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} +$ &c.. Per tradita ergo præcepta ſummari poterunt hæ Series

$$1 \pm \frac{1}{2^n} + \frac{1}{3^n} \pm \frac{1}{4^n} + \frac{1}{5^n} \pm \frac{1}{6^n} + \frac{1}{7^n} \pm \text{\&c.}$$
$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{\&c.}$$

Si quidem n ſit numerus par, atque ſumma erit $= A\pi^n$ exiſtente A numero rationali.

171. Præterea vero expreſſiones §. 164 exhibitæ ſimili modo

LIB. I. Series notatu dignas ſuppeditabunt. Cum enim ſit $cof. \frac{1}{2} v + tang. \frac{1}{2} g. \; ſin. \frac{1}{2} v = (1 + \frac{v}{\pi - g})(1 - \frac{v}{\pi + g})(1 + \frac{v}{3\pi - g})$ &c., ſi ponamus $v = \frac{x}{n}\pi$ & $g = \frac{m}{n}\pi$ erit $(1 + \frac{x}{n - m})(1 - \frac{x}{n + m})(1 + \frac{x}{3n - m})(1 - \frac{x}{3n + m})(1 + \frac{x}{5n - m})(1 - \frac{x}{5n + m})$ &c. $= cof. \frac{x\pi}{2n} + tang. \frac{m\pi}{2n}.$ $ſin. \frac{x\pi}{2n} = 1 + \frac{\pi x}{2n} \; tang. \frac{m\pi}{2n} - \frac{\pi\pi xx}{2.4nn} - \frac{\pi^3 x^3}{2.4.6n^3} \; tang. \frac{m\pi}{2n} + \frac{\pi^4 x^4}{2.4.6.8n^4} +$ &c. Hæc expreſſio infinita cum §. 165 collata dabit hos valores $A = \frac{\pi}{2n} \; tang. \frac{m\pi}{2n}$; $B = \frac{-\pi\pi}{2.4nn}$; $C = \frac{-\pi^3}{2.4.6n^3}. \; tang. \frac{m\pi}{2n}$; $D = \frac{\pi^4}{2.4.6.8n^4}$; $E = \frac{\pi^5}{2.4.6.8.10n^5}. \; tang. \frac{m\pi}{2n}$ &c. Tum vero erit $\alpha = \frac{1}{n - m}$; $\beta = -\frac{1}{n + m}$; $\gamma = \frac{1}{3n - m}$; $\delta = -\frac{1}{3n + m}$; $\epsilon = \frac{1}{5n - m}$; $\zeta = -\frac{1}{5n + m}$ &c.

172. Hinc ergo ad normam §. 166 ſequentes Series exorientur.

$$P = \frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \frac{1}{5n-m} - \frac{1}{5n+m} + \&c.$$

$$Q = \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(3n-m)^2} + \frac{1}{(3n+m)^2} + \frac{1}{(5n-m)^2} + \&c.$$

$$R = \frac{1}{(n-m)^3} - \frac{1}{(n+m)^3} + \frac{1}{(3n-m)^3} - \frac{1}{(3n+m)^3} + \frac{1}{(5n-m)^3} - \&c.$$

$$S = \frac{1}{(n-m)^4} + \frac{1}{(n+m)^4} + \frac{1}{(3n-m)^4} + \frac{1}{(3n+m)^4} + \frac{1}{(5n-m)^4} + \&c.$$

$$T = \frac{1}{(n-m)^5} - \frac{1}{(n+m)^5} + \frac{1}{(3n-m)^5} - \frac{1}{(3n+m)^5} + \frac{1}{(5n-m)^5} - \&c.$$

$$V = \frac{1}{(n-m)^6} + \frac{1}{(n+m)^6} + \frac{1}{(3n-m)^6} + \frac{1}{(3n+m)^6} + \frac{1}{(5n-m)^6} + \&c.$$

&c.

Poſito autem $tang.\ \frac{m\pi}{2n} = k$ erit, uti oſtendimus,

$$P = A = \frac{k\pi}{2n}$$

$$Q = \frac{(kk+1)\pi\pi}{4nn} = \frac{(2kk+2)\pi^2}{2.4nn}$$

$$R = \frac{(k^3+k)\pi^3}{8n^3} = \frac{(6k^3+6k)\pi^3}{2.4.6n^3}$$

$$S = \frac{(3k^4+4kk+1)\pi^4}{48n^4} = \frac{(24k^4+32k^2+8)\pi^4}{2.4.6.8n^4}$$

$$T = \frac{(3k^5+5k^3+2k)\pi^5}{96n^5} = \frac{(120k^5+200k^3+80k)\pi^5}{2.4.6.8.10n^5}$$

&c.

173. Pari modo ultima forma §. 164, $coſ.\ \frac{1}{2}v + cot.\ \frac{1}{2}g \times ſin.\ \frac{1}{2}v = (1+\frac{v}{g})(1-\frac{v}{2\pi-g})(1+\frac{v}{2\pi+g})(1-\frac{v}{4\pi-g})(1+\frac{v}{4\pi+g})$ &c., ſi ponamus $v = \frac{x}{n}\pi$, $g = \frac{m}{n}\pi$ & $tang.\ \frac{m\pi}{2n} = k$, ut ſit $cot.\ \frac{1}{2}g = \frac{1}{k}$, dabit $coſ.\ \frac{\pi x}{2n} + \frac{1}{k} \times ſin.\ \frac{\pi x}{2n} = 1 + \frac{\pi x}{2nk} - \frac{\pi\pi xx}{2.4nn} - \frac{\pi^3 x^3}{2.4.6n^3k} + \frac{\pi^4 x^4}{2.4.6.8n^4} + \frac{\pi^5 x^5}{2.4.6.8.10n^5k} - \&c. = (1+\frac{x}{m})(1-\frac{x}{2n-m})(1+\frac{x}{2n+m})$

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 $(1 - \frac{x}{4n - m})(1 + \frac{x}{4n + m},)$ &c.. Comparatione ergo cum forma generali (§. 165) inſtituta erit $A = \frac{\pi}{2nk}$; $B = \frac{-\pi\pi}{2.4n^2}$; $C = \frac{-\pi^3}{2.4.6n^3k}$; $D = \frac{\pi^4}{2.4.6.8n^4}$; $E = \frac{\pi^5}{2.4.6.8.10n^5k}$; &c.; ex Factoribus vero habebitur $\alpha = \frac{1}{m}$; $\beta = \frac{-1}{2n - m}$; $\gamma = \frac{1}{2n + m}$; $\delta = \frac{-1}{4n - m}$; $\epsilon = \frac{1}{4n + m}$ &c.

174. Hinc ergo ad normam §. 166 ſequentes Series formabuntur, earumque ſummæ aſſignabuntur

$$P = \frac{1}{m} - \frac{1}{2n - m} + \frac{1}{2n + m} - \frac{1}{4n - m} + \frac{1}{4n + m} - \&c.$$

$$Q = \frac{1}{m^2} + \frac{1}{(2n - m)^2} + \frac{1}{(2n + m)^2} + \frac{1}{(4n - m)^2} + \frac{1}{(4n + m)^2} + \&c.$$

$$R = \frac{1}{m^3} - \frac{1}{(2n - m)^3} + \frac{1}{(2n + m)^3} - \frac{1}{(4n - m)^3} + \frac{1}{(4n + m)^3} - \&c.$$

$$S = \frac{1}{m^4} + \frac{1}{(2n - m)^4} + \frac{1}{(2n + m)^4} + \frac{1}{(4n - m)^4} + \frac{1}{(4n + m)^4} + \&c.$$

$$T = \frac{1}{m^5} - \frac{1}{(2n - m)^5} + \frac{1}{(2n + m)^5} - \frac{1}{(4n - m)^5} + \frac{1}{(4n + m)^5} - \&c.$$

&c.

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Hæ autem ſummæ P, Q, R, S, &c. ita ſe habebunt

$$P = A = \frac{\pi}{2nk} = \frac{1\pi}{2nk}$$
$$Q = \frac{(kk+1)\pi\pi}{4nnkk} = \frac{(2+2kk)\pi^2}{2.4n^2k^2}$$
$$R = \frac{(kk+1)\pi^3}{8n^3k^3} = \frac{(6+6kk)\pi^3}{2.4.6n^3k^3}$$
$$S = \frac{(k^4+4kk+3)\pi^4}{48n^4k^4} = \frac{(24+32kk+3k^4)\pi^4}{2.4.6.8n^4k^4}$$
$$T = \frac{(2k^4+5kk+3)\pi^5}{96n^5k^5} = \frac{(120+200kk+80k^4)\pi^5}{2.4.6.8.10n^5k^5}$$
$$V = \frac{(2k^6+17k^4+30k^2+15)\pi^6}{960n^6k^6} = \frac{(720+1440kk+816k^4+96k^6)\pi^6}{2.4.6.8.10.12n^6k^6}$$

&c.

175. Series iſtæ generales merentur ut caſus quoſdam particulares inde derivemus, qui prodibunt ſi rationem m ad n in numeris determinemus. Sit igitur primum $m = 1$ & $n = 2$, fiet $k = tang.\ \frac{\pi}{4} = tang.\ 45^{\circ} = 1$, atque ambæ Serierum claſſes inter ſe congruent. Erit ergo

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$$
$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$$
$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \&c.$$
$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.$$
$$\frac{5\pi^5}{1536} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \&c.$$
$$\frac{\pi^6}{960} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.$$

&c.

Harum Serierum primam jam ſupra (§. 140) elicuimus, reliquarum illæ, quæ pares habent Dignitates, modo ante (§. 169) ſunt erutæ; ceteræ, in quibus Exponentes ſunt numeri impa-

 res, hic primum occurrunt. Conſtat ergo omnium quoque iſtarum Serierum:

$$1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \frac{1}{9^{2n+1}} - \&c.$$

ſummas per valorem ipſius π aſſignari poſſe.

176. Sit nunc $m = 1$, $n = 3$; erit $k = tang. \frac{\pi}{6} = tang. 30^{\circ} = \frac{1}{\sqrt{3}}$; atque Series §. 172 abibunt in has

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \&c.$$

$$\frac{\pi\pi}{27} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{16^2} + \&c.$$

$$\frac{\pi^3}{162\sqrt{3}} = \frac{1}{2^3} - \frac{1}{4^3} + \frac{1}{8^3} - \frac{1}{10^3} + \frac{1}{14^3} - \frac{1}{16^3} + \&c.$$

&c., ſive

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \&c.$$

$$\frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \&c.$$

$$\frac{4\pi^3}{81\sqrt{3}} = 1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \&c.$$

&c.

in his Seriebus deſunt omnes numeri per ternarium diviſibiles: hinc pares dimenſiones ex jam inventis deducentur hoc modo. Cum ſit

$$\frac{\pi\pi}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c., \text{ erit}$$

$$\frac{\pi\pi}{6.9} = \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{12^2} + \&c. = \frac{\pi\pi}{54},$$

quæ poſterior Series continens omnes numeros per ternarium diviſibiles, ſi ſubtrahatur a priore, remanebunt omnes numeri non diviſibiles per 3: ſicque erit $\frac{8\pi\pi}{54} = \frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.$, uti invenimus.

177. Eadem hypotheſis $m = 1$, $n = 3$ & $k = \frac{1}{\sqrt{3}}$, ad §. 174 accommodata has præbebit ſummationes

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \&c.$$
$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \&c.$$
$$\frac{\pi^3}{18\sqrt{3}} = 1 - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^3} - \&c.$$
&c.

in quarum denominatoribus numeri tantum impares occurrunt exceptis iis, qui per ternarium ſunt diviſibiles. Ceterum pares dimenſiones ex jam cognitis deduci poſſunt, cum enim ſit

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.,\ \text{erit}$$
$$\frac{\pi\pi}{8.9} = \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \&c. = \frac{\pi\pi}{72}$$

quæ Series, omnes numeros impares per 3 diviſibiles continens, ſi ſubtrahatur a ſuperiore, relinquet Seriem quadratorum numerorum imparium per 3 non diviſibilium, eritque

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \&c.$$

178. Si Series in §. §. 172 & 174 inventæ vel addantur vel ſubtrahantur, obtinebuntur aliæ Series notatu dignæ. Erit ſcilicet $\frac{k\pi}{2n} + \frac{\pi}{2nk} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \&c. = \frac{(kk+1)\pi}{2nk}$: at eſt $k = tang.\ \frac{m\pi}{2n} = \frac{ſin.\ \frac{m\pi}{2n}}{coſ.\ \frac{m\pi}{2n}}$, & $1 + kk = \frac{1}{(coſ.\ \frac{m\pi}{2n})^2}$, unde $\frac{2k}{1+kk} = 2\,ſin.\ \frac{m\pi}{2n} \times coſ.\ \frac{m\pi}{2n} = ſin.\ \frac{m\pi}{n}$, quo valore ſubſtituto habebimus

$$\frac{\pi}{n\,ſin.\ \frac{m\pi}{n}} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} +$$

LIB. I. $\frac{1}{3n-m} - \frac{1}{3n+m} -$ &c. . Simili modo per ſubtractionem erit $\frac{\pi}{2nk} - \frac{k\pi}{2n} = \frac{(1-kk)\pi}{2nk} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \frac{1}{3n+m} -$ &c., at eſt $\frac{2k}{1-kk} = tang.\ 2.\ \frac{m\pi}{2n} = tang.\ \frac{m\pi}{n} = \frac{ſin.\ \frac{m\pi}{n}}{coſ.\ \frac{m\pi}{n}}$, hinc erit $\frac{\pi.\ coſ.\frac{m\pi}{n}}{n.\ ſin.\ \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} +$ &c. . Series Quadratorum & altiorum Poteſtatum hinc ortæ facilius per differentiationem hinc deducentur infra.

179. Quoniam caſus, quibus $m = 1$ & $n = 2$ vel 3, jam evolvimus, ponamus $m = 1$ & $n = 4$; erit $ſin.\ \frac{m\pi}{n} = ſin.\ \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ & $coſ.\ \frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Hinc itaque habebitur

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{\&c.}$$

$$\text{\&}\ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{\&c.}$$

Sit $m = 1$ & $n = 8$, erit $\frac{m\pi}{n} = \frac{\pi}{8}$ & $ſin.\ \frac{\pi}{8} = \sqrt{(\frac{1}{2} - \frac{1}{2\sqrt{2}})}$ & $coſ.\ \frac{\pi}{8} = \sqrt{(\frac{1}{2} + \frac{1}{2\sqrt{2}})}$ & $\frac{coſ.\ \frac{\pi}{8}}{ſin.\ \frac{\pi}{8}} = 1 + \sqrt{2}$. Hinc itaque erit (178) per primam combinationem

$$\frac{\pi}{4\sqrt{(2-\sqrt{2})}} = 1 + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} + \frac{1}{23} - \text{\&c.}$$

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& per ſecundam, quoniam $1 + \sqrt{2} = \frac{1}{-1+\sqrt{2}}$,

$$\frac{\pi}{8(-1+\sqrt{2})} = 1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \&c.$$

Sit nunc $m = 3$ & $n = 8$, erit $\frac{m\pi}{n} = \frac{3\pi}{8}$ & $\mathit{ſin.}\ \frac{3\pi}{8} = \sqrt{(\frac{1}{2} + \frac{1}{2\sqrt{2}})}$, & $\mathit{cof.}\ \frac{3\pi}{8} = \sqrt{(\frac{1}{2} - \frac{1}{2\sqrt{2}})}$, unde $\frac{\mathit{cof.}\ \frac{3}{8}\pi}{\mathit{ſin.}\ \frac{3}{8}\pi} = \frac{1}{1+\sqrt{2}}$; ac prodibunt hæ Series

$$\frac{\pi}{4\sqrt{(2+\sqrt{2})}} = \frac{1}{3} + \frac{1}{5} - \frac{1}{11} - \frac{1}{13} + \frac{1}{19} + \frac{1}{21} - \&c.$$

$$\frac{\pi}{8(1+\sqrt{2})} = \frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \&c.$$

180. Ex his ſeriebus naſcuntur,

1°. Addendo primam tertiæ

$$\frac{\pi\sqrt{(2+\sqrt{2})}}{4} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

2°. Subtrahendo tertiam a prima

$$\frac{\pi\sqrt{(2-\sqrt{2})}}{4} = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \&c.$$

3°. Addendo quartæ primam

$$\frac{\pi(-1+\sqrt{2}+\sqrt{(4+2\sqrt{2})})}{8} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

4°. Subtrahendo quartam a prima

$$\frac{\pi(1-\sqrt{2}+\sqrt{(4+2\sqrt{2})})}{8} = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \&c.$$

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5°. Addendo ſecundam tertiæ

$$\frac{\pi(1+\sqrt{2}+\sqrt{(4-2\sqrt{2})})}{8} = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

6°. Subtrahendo tertiam a ſecunda

$$\frac{\pi(1+\sqrt{2}-\sqrt{(4-2\sqrt{2})})}{8} = 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} - \&c.$$

Simili modo, ponendo $n = 16$ & m vel 1 vel 3 vel 5 vel 7, ulterius progredi licet, hocque modo ſummæ reperientur Serierum $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$, &c., in quibus ſignorum + & — viciſſitudines alias leges ſequantur.

181. Si in Seriebus §. 178 inventis bini termini in unam ſummam colligantur, erit

$$\frac{\pi}{n.\,fin.\,\frac{m\pi}{n}} = \frac{1}{m} + \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} + \frac{2m}{9nn-mm} - \frac{2m}{16nn-mm} + \&c,$$

ideoque

$$\frac{1}{nn-mm} - \frac{1}{4nn-mm} + \frac{1}{9nn-mm} - \&c. = \frac{\pi}{2mn.\,fin.\,\frac{m\pi}{n}} - \frac{1}{2mm}.$$

Altera vero Series dabit

$$\frac{\pi}{n.\,tang.\,\frac{m}{n}\pi} = \frac{1}{m} - \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} - \frac{2m}{9nn-mm} - \&c.$$

hincque

$$\frac{1}{nn-mm}+\frac{1}{4nn-mm}+\frac{1}{9nn-mm}+\&c. = \frac{1}{2mm}-\frac{\pi}{2mn.\ tang.\frac{m}{n}\pi}.$$

Ex his autem conjunctis naſcitur hæc

$$\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\&c. = \frac{\pi.\ tang.\frac{m}{2n}\pi}{4mn}.$$

Si in hac Serie ſit $n=1$ & m numerus par quicunque $=2k$, ob $tang.\ k\pi = 0$, erit ſemper, niſi ſit $k=0$,

$$\frac{1}{1-4kk}+\frac{1}{9-4kk}+\frac{1}{25-4kk}+\frac{1}{49-4kk}+\&c. = 0,$$

ſin autem in illa Serie fiat $n=2$ & m fuerit numerus quicunque impar $=2k+1$, ob $\frac{1}{tang.\frac{m\pi}{n}}=0$, erit $\frac{1}{4-(2k+1)^2}+\frac{1}{16-(2k+1)^2}+\frac{1}{36-(2k+1)^2}+\&c. = \frac{1}{2(2k+1)^2}$.

182. Multiplicentur Series inventæ per nn ſitque $\frac{m}{n}=p$, habebuntur iſtæ formæ

$$\frac{1}{1-pp}-\frac{1}{4-pp}+\frac{1}{9-pp}-\frac{1}{16-pp}+\&c. = \frac{\pi}{2p.\ ſin.\ p\pi}-\frac{1}{2pp}.$$

$$\frac{1}{1-pp}+\frac{1}{4-pp}+\frac{1}{9-pp}+\frac{1}{16-pp}+\&c. = \frac{1}{2pp}-\frac{\pi}{2p.\ ſin.\ p\pi}.$$

Sit $pp=a$, atque naſcentur hæ Series

$$\frac{1}{1-a}-\frac{1}{4-a}+\frac{1}{9-a}-\frac{1}{16-a}+\&c. = \frac{\pi\sqrt{a}}{2a.\ ſin.\ \pi\sqrt{a}}-\frac{1}{2a}$$

$$\frac{1}{1-a}+\frac{1}{4-a}+\frac{1}{9-a}+\frac{1}{16-a}+\&c. = \frac{1}{2a}-\frac{\pi\sqrt{a}}{2a.\ tang.\ \pi\sqrt{a}}:$$

Dummodo ergo a non fuerit numerus negativus nec quadratus integer, ſumma harum Serierum per Circulum exhiberi poterit.

183. Per reductionem autem exponentialium imaginariorum ad Sinus & Coſinus Arcuum circularium ſupra traditam poterimus quoque ſummas harum Serierum aſſignare ſi a ſit numerus negativus. Cum enim ſit $e^{x\sqrt{-1}} = cof.\, x + \sqrt{-1} \times fin.\, x$ & $e^{-x\sqrt{-1}} = cof.\, x - \sqrt{-1}.\, fin.\, x$, erit viciſſim, poſito $y\sqrt{-1}$ loco x; $cof.\, y\sqrt{-1} = \frac{e^{-y} + e^{y}}{2}$ & $fin.\, y\sqrt{-1} = \frac{e^{-y} - e^{y}}{2\sqrt{-1}}$. Quod ſi ergo $a = -b$ & $y = \pi\sqrt{b}$, erit $cof.\, \pi\sqrt{-b} = \frac{e^{-\pi\sqrt{b}} + e^{\pi\sqrt{b}}}{2}$ & $fin.\, \pi\sqrt{-b} = \frac{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}{2\sqrt{-1}}$; ideoque $tang.\, \pi\sqrt{-b} = \frac{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}{(e^{-\pi\sqrt{b}} + e^{\pi\sqrt{b}})\sqrt{-1}}$. Hinc erit $\frac{\pi\sqrt{-b}}{fin.\, \pi\sqrt{-b}} = \frac{-2\pi\sqrt{b}}{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}$; & $\frac{\pi\sqrt{-b}}{tang.\, \pi\sqrt{-b}} = \frac{(e^{-\pi\sqrt{b}} + e^{\pi\sqrt{b}})\pi\sqrt{b}}{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}$. His ergo notatis, erit

$$\frac{1}{1+b} - \frac{1}{4+b} + \frac{1}{9+b} - \frac{1}{16+b} + \&c. = \frac{1}{2b} - \frac{\pi\sqrt{b}}{(e^{\pi\sqrt{b}} - e^{-\pi\sqrt{b}})b},$$

$$\frac{1}{1+b} + \frac{1}{4+b} + \frac{1}{9+b} + \frac{1}{16+b} + \&c. = \frac{(e^{\pi\sqrt{b}} + e^{-\pi\sqrt{b}})\pi\sqrt{b}}{2b(e^{\pi\sqrt{b}} - e^{-\pi\sqrt{b}})} - \frac{1}{2b}.$$

Eædem hæ Series deduci poſſunt ex §. 162 adhibendo eandem methodum, qua in

in hoc capite ſum uſus. Quoniam vero hoc pacto reductio Sinuum & Coſinuum Arcuum imaginariorum ad quantitates exponentiales reales, non mediocriter illuſtratur, hanc explicationem alteri præferendam duxi. CAP. X.

CAPUT XI.

De aliis Arcuum atque Sinuum expreſſionibus infinitis.

184. QUONIAM ſupra (158.) denotante z Arcum Circuli quemcunque, vidimus eſſe $\sin. z = z\left(1 - \frac{zz}{\pi\pi}\right)\left(1 - \frac{zz}{4\pi\pi}\right)\left(1 - \frac{zz}{9\pi\pi}\right)\left(1 - \frac{zz}{16\pi\pi}\right)$ &c., & $\cos. z = \left(1 - \frac{4zz}{\pi\pi}\right)\left(1 - \frac{4zz}{9\pi\pi}\right)\left(1 - \frac{4zz}{25\pi\pi}\right)\left(1 - \frac{4zz}{49\pi\pi}\right)$ &c., ponamus eſſe Arcum $z = \frac{m\pi}{n}$, erit $\sin. \frac{m\pi}{n} = \frac{m\pi}{n}\left(1 - \frac{mm}{nn}\right)\left(1 - \frac{mm}{4nn}\right)\left(1 - \frac{mm}{9nn}\right)\left(1 - \frac{mm}{16nn}\right)$ &c., & $\cos. \frac{m}{n}\pi = \left(1 - \frac{4mm}{nn}\right)\left(1 - \frac{4mm}{9nn}\right)\left(1 - \frac{4mm}{25nn}\right)\left(1 - \frac{4mm}{49nn}\right)$ &c. Vel ponatur $2n$ loco n, ut prodeant hæ expreſſiones:

$$\sin. \frac{m\pi}{2n} = \frac{m\pi}{2n}\left(\frac{4nn - mm}{4nn}\right)\left(\frac{16nn - mm}{16nn}\right)\left(\frac{36nn - mm}{36nn}\right) \&c.$$

$$\cos. \frac{m\pi}{2n} = \left(\frac{nn - mm}{nn}\right)\left(\frac{9nn - mm}{9nn}\right)\left(\frac{25nn - mm}{25nn}\right)\left(\frac{49nn - mm}{49nn}\right) \&c.,$$

quæ, in Factores ſimplices reſolutæ, dant

$$\sin. \frac{m\pi}{2n} = \frac{m\pi}{2n}\left(\frac{2n - m}{2n}\right)\left(\frac{2n + m}{2n}\right)\left(\frac{4n - m}{4n}\right)\left(\frac{4n + m}{4n}\right)\left(\frac{6n - m}{6n}\right) \&c.$$

LIB. I. $cof. \frac{m\pi}{2n} = (\frac{n-m}{n})(\frac{n+m}{n})(\frac{3n-m}{3n})(\frac{3n+m}{3n})(\frac{5n-m}{5n})$
$(\frac{5n+m}{5n})$ &c.

Ponatur $n-m$ loco m, quia eſt $\mathit{ſin}. \frac{(n-m)\pi}{2n} = cof. \frac{m\pi}{2n}$ & $cof. \frac{(n-m)\pi}{2n} = \mathit{ſin}. \frac{m\pi}{2n}$, provenient hæ expreſſiones :

$$cof. \frac{m\pi}{2n} = (\frac{(n-m)\pi}{2n})(\frac{n+m}{2n})(\frac{3n-m}{2n})(\frac{3n+m}{4n})(\frac{5n-m}{4n})(\frac{5n+m}{6n}) \text{ \&c.}$$

$$\mathit{ſin}. \frac{m\pi}{2n} = \frac{m}{n}(\frac{2n-m}{n})(\frac{2n+m}{3n})\frac{4n-m}{3n})(\frac{4n+m}{5n})(\frac{6n-m}{5n}) \text{ \&c.}$$

185. Cum igitur pro Sinu & Coſinu Anguli $\frac{m\pi}{2n}$ binæ habeantur expreſſiones, ſi eæ inter ſe comparentur dividendo, erit $1 = \frac{\pi}{2} . \frac{1}{2} . \frac{3}{2} . \frac{3}{4} . \frac{5}{4} . \frac{5}{6} . \frac{7}{6} . \frac{7}{8} . \frac{9}{8}$. &c., ideoque $\frac{\pi}{2} = \frac{2.2.4.4.6.6.8.8.10.10.12.12.}{1.3.3.5.5.7.7.9.9.11.11.13.}$ &c., quæ eſt expreſſio pro Peripheria Circuli, quam WALLISIUS invenit in *Arithmetica infinitorum*. Similes autem huic innumeras expreſſiones exhibere licet ope primæ expreſſionis pro Sinu ; ex ea enim deducitur fore :

$$\frac{\pi}{2} = \frac{n}{m}.\mathit{ſin}. \frac{m\pi}{2n}(\frac{2n}{2n-m})(\frac{2n}{2n+m})(\frac{4n}{4n-m})(\frac{4n}{4n+m})(\frac{6n}{6n-m}) \text{ \&c.},$$

quæ, poſito $\frac{m}{n} = 1$, præbet illam ipſam WALLISII formulam.

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Sit ergo $\frac{m}{n} = \frac{1}{2}$, ob $ſin. \frac{1}{4}\pi = \frac{1}{\sqrt{2}}$, erit

$\frac{\pi}{2} = \frac{\sqrt{2}}{1} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{16}{15} \cdot \frac{16}{17} \cdot$ &c.

Sit $\frac{m}{n} = \frac{1}{3}$, ob $ſin. \frac{1}{6}\pi = \frac{1}{2}$, erit

$\frac{\pi}{2} = \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{12}{13} \cdot \frac{18}{17} \cdot \frac{18}{19} \cdot \frac{24}{23}$ &c..

Quod ſi Expreſſio *Wallisiana* dividatur per illam ubi $\frac{m}{n} = \frac{1}{2}$,

erit $\sqrt{2} = \frac{2.2.6.6.10.10.14.14.18.18.}{1.3.5.7.9.11.13.15.17.19.}$ &c.

186. Quoniam Tangens cujuſque Anguli æquatur Sinui per Coſinum diviſo, Tangens quoque per hujuſmodi Factores infinitos exprimi poterit. Quod ſi autem prima Sinus expreſſio dividatur per alteram Coſinus expreſſionem, erit

$tang. \frac{m\pi}{2n} = \frac{m}{n-m} \left(\frac{2n-m}{n+m}\right) \left(\frac{2n+m}{3n-m}\right) \left(\frac{4n-m}{3n+m}\right) \left(\frac{4n+m}{5n-m}\right)$ &c., &

$cot. \frac{m\pi}{2n} = \frac{n-m}{m} \left(\frac{n+m}{2n-m}\right) \left(\frac{3n-m}{2n+m}\right) \left(\frac{3n+m}{4n-m}\right) \left(\frac{5n-m}{4n+m}\right)$ &c..

Simili modo autem Secantes & Coſecantes exprimentur

$ſec. \frac{m\pi}{2n} = \left(\frac{n}{n-m}\right) \left(\frac{n}{n+m}\right) \left(\frac{3n}{3n-m}\right) \left(\frac{3n}{3n+m}\right) \left(\frac{5n}{5n-m}\right) \left(\frac{5n}{5n+m}\right)$ &c.

$coſec. \frac{m\pi}{2n} = \frac{n}{m} \left(\frac{n}{2n-m}\right) \left(\frac{3n}{2n+m}\right) \left(\frac{3n}{4n-m}\right) \left(\frac{5n}{4n+m}\right) \left(\frac{5n}{6n-m}\right)$ &c.

Sin autem alteræ Sinuum & Coſinuum formulæ combinentur, erit

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$$tang.\frac{m\pi}{2n} = \frac{\pi}{2}\cdot\frac{m}{n-m}\cdot\frac{1(2n-m)}{2(n+m)}\cdot\frac{3(2n+m)}{2(3n-m)}\cdot\frac{3(4n-m)}{4(3n+m)}\cdot\&c.$$

$$cot.\ \frac{m\pi}{2n} = \frac{\pi}{2}\cdot\frac{n-m}{m}\cdot\frac{1(n+m)}{2(2n-m)}\cdot\frac{3(3n-m)}{2(2n+m)}\cdot\frac{3(3n+m)}{4(4n-m)}\cdot\&c.$$

$$ſec.\ \frac{m\pi}{2n} = \frac{2}{\pi}\cdot\frac{n}{n-m}\cdot\frac{2n}{n+m}\cdot\frac{2n}{3n-m}\cdot\frac{4n}{3n+m}\cdot\frac{4n}{5n-m}\cdot\&c.$$

$$coſec.\frac{m\pi}{2n} = \frac{2}{\pi}\cdot\frac{n}{m}\cdot\frac{2n}{2n-m}\cdot\frac{2n}{2n+m}\cdot\frac{4n}{4n-m}\cdot\frac{4n}{4n+m}\cdot\&c.$$

187. Si loco m ſcribatur k, ſimilique modo Anguli $\frac{k\pi}{2n}$ Sinus & Coſinus definiantur, ac per has expreſſiones illæ priores dividantur, prodibunt iſtæ formulæ

$$\frac{ſin.\frac{m\pi}{2n}}{ſin.\frac{k\pi}{2n}} = \frac{m}{k}\cdot\frac{2n-m}{2n-k}\cdot\frac{2n+m}{2n+k}\cdot\frac{4n-m}{4n-k}\cdot\frac{4n+m}{4n+k}\cdot\&c.$$

$$\frac{ſin.\frac{m\pi}{2n}}{coſ.\frac{k\pi}{2n}} = \frac{m}{n-k}\cdot\frac{2n-m}{n+k}\cdot\frac{2n+m}{3n-k}\cdot\frac{4n-m}{3n+k}\cdot\frac{4n+m}{5n-k}\cdot\&c.$$

$$\frac{coſ.\frac{m\pi}{2n}}{coſ.\frac{k\pi}{2n}} = \left(\frac{n-m}{n-k}\right)\left(\frac{n+m}{n+k}\right)\left(\frac{3n-m}{3n-k}\right)\left(\frac{3n+m}{3n+k}\right)\left(\frac{5n-m}{5n-k}\right).\ \&c.$$

$$\frac{coſ.\frac{m\pi}{2n}}{ſin.\frac{k\pi}{2n}} = \left(\frac{n-m}{k}\right)\left(\frac{n+m}{2n-k}\right)\left(\frac{3n-m}{2n+k}\right)\left(\frac{3n+m}{4n-k}\right)\left(\frac{5n-m}{4n+k}\right).\ \&c.$$

Sumto ergo pro $\frac{k\pi}{2n}$ ejuſmodi Angulo cujus Sinus & Coſinus dentur, per hos licebit alius cujuſcunque Anguli $\frac{m\pi}{2n}$ Sinum & Coſinum determinare.

188. Viciſſim igitur hujuſmodi expreſſionum, quæ ex Factoribus infinitis conſtant, valores veri vel per Circuli Peri-

phериam, vel per Sinus & Coſinus Angulorum datorum aſſignari poſſunt, quod ipſum non parvi eſt momenti, cum etiamnunc aliæ methodi non conſtent, quarum ope hujuſmodi productorum infinitorum valores exhiberi queant. Ceterum vero hujuſmodi expreſſiones parum utilitatis afferunt, ad valores cum ipſius π tum Sinuum Coſinuumve Angulorum $\frac{m\pi}{2n}$ per approximationem eruendos. Quanquam enim iſti Factores $\frac{\pi}{2} = 2(1-\frac{1}{9})(1-\frac{1}{25})(1-\frac{1}{49})$ &c., in fractionibus decimalibus non difficulter in ſe multiplicantur, tamen nimis multi termini in computum duci deberent, ſi valorem ipſius π ad decem tantum figuras juſtum invenire vellemus.

189. Præcipuus autem uſus hujuſmodi expreſſionum, etſi infinitarum, in inventione Logarithmorum verſatur, in quo negotio Factorum utilitas tanta eſt, ut ſine illis Logarithmorum ſupputatio eſſet difficillima. Ac primo quidem, cum ſit $\pi = 4(1-\frac{1}{9})(1-\frac{1}{25})(1-\frac{1}{49})$ &c., erit, ſumendis Logarithmis, $l\pi = l4 + l(1-\frac{1}{9}) + l(1-\frac{1}{25}) + l(1-\frac{1}{49}) +$ &c., vel $l\pi = l2 - l(1-\frac{1}{4}) - l(1-\frac{1}{16}) - l(1-\frac{1}{36})$ — &c., ſive Logarithmi communes ſive hyperbolici ſumantur. Quoniam vero ex Logarithmis hyperbolicis vulgares facile reperiuntur, inſigne compendium adhiberi poterit ad Logarithmum hyperbolicum ipſius π inveniendum.

190. Cum igitur, Logarithmis hyperbolicis ſumendis, ſit $l(1-x) = -x - \frac{xx}{2} - \frac{x^3}{3} - \frac{x^4}{4} -$ &c., ſi hoc modo ſinguli termini evolvantur, erit

$$l\pi = l4 \begin{cases} -\frac{1}{9} - \frac{1}{2.\ 9^2} - \frac{1}{3.\ 9^3} - \frac{1}{4.\ 9^4} - \&c. \\ -\frac{1}{25} - \frac{1}{2.\ 25^2} - \frac{1}{3.\ 25^3} - \frac{1}{4.\ 25^4} - \&c. \\ -\frac{1}{49} - \frac{1}{2.\ 49^2} - \frac{1}{3.\ 49^3} - \frac{1}{4.\ 49^4} - \&c. \end{cases}$$
&c.

In his Seriebus numero infinitis verticaliter deſcendendo ejuſmodi prodeunt Series, quarum ſummas ſupra jam invenimus, quare ſi brevitatis gratia ponamus

$$A = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$$
$$B = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.$$
$$C = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.$$
$$D = 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \&c.$$
&c.

erit $l\pi = l4 - (A - 1) - \frac{1}{2}(B - 1) - \frac{1}{3}(C - 1) - \frac{1}{4}(D - 1) - \&c.$

Eſt vero, ſummis ſupra inventis proxime exprimendis,

$$\begin{aligned} A &= 1,2337005501361698273543 1 \\ B &= 1,0146780316041920545462 5 \\ C &= 1,0014470766409421219064 7 \\ D &= 1,0001551790252961193029 8 \\ E &= 1,0000170413630448255081 6 \\ F &= 1,0000018858485831195759 0 \\ G &= 1,0000002092405192115001 0 \\ H &= 1,0000000232371573791567 0 \end{aligned}$$

$I = 1,00000000258143755665977$
$K = 1,00000000028680769745558$
$L = 1,00000000003186677514044$
$M = 1,00000000000354072294392$
$N = 1,00000000000039341246691$
$O = 1,00000000000004371244859$
$P = 1,00000000000000485693682$
$Q = 1,00000000000000053965957$
$K = 1,00000000000000005996217$
$S = 1,00000000000000000666246$
$T = 1,00000000000000000074027$
$V = 1,00000000000000000008225$
$W = 1,00000000000000000000913$
$X = 1,00000000000000000000101$

Hinc ſine tædioſo calculo reperitur Logarithmus hyperbolicus ipſius $\pi = 1,14472988584940017414342$, qui ſi multiplicetur per $0,43429$ &c., prodit Logarithmus vulgaris ipſius $\pi = 0,497149872694133854351 26$.

191. Quia porro tam Sinum quam Coſinum Anguli $\frac{m\pi}{2n}$ expreſſum habemus per Factores numero infinitos, utriuſque Logarithmum commode exprimere poterimus. Erit autem ex formulis primo inventis

$$l\,\mathit{ſin}.\,\frac{m\pi}{2n} = l\pi + l\frac{m}{2n} + l\left(1 - \frac{mm}{4nn}\right) + l\left(1 - \frac{mm}{16nn}\right) + l\left(1 - \frac{mm}{36nn}\right)\ \&c.$$

$$l\,\mathit{coſ}.\,\frac{m\pi}{2n} = l\left(1 - \frac{mm}{nn}\right) + l\left(1 - \frac{mm}{9nn}\right) + l\left(1 - \frac{mm}{25nn}\right) + l\left(1 - \frac{mm}{49nn}\right)\ \&c.$$

Hinc primum Logarithmi hyperbolici, ut ante, per Series maxime convergentes facile exprimuntur. Ne autem præter

 neceſſitatem Series infinitas multiplicemus, terminos priores actu in Logarithmis involutos relinquamus, eritque

$$l\,ſin.\frac{m\pi}{2n} = l\pi + lm + l(2n-m) + l(2n+m) - l8 - 3ln$$
$$- \frac{mm}{16nn} - \frac{m^4}{2.\,16^2n^4} - \frac{m^6}{3.\,16^3n^6} - \frac{m^8}{4.\,16^4n^8} - \&c.$$
$$- \frac{mm}{36nn} - \frac{m^4}{2.\,36^2n^4} - \frac{m^6}{3.\,36^3n^6} - \frac{m^8}{4.\,36^4n^8} - \&c.$$
$$- \frac{mm}{64nn} - \frac{m^4}{2.\,64^2n^4} - \frac{m^6}{3.\,64^3n^6} - \frac{m^8}{4.\,64^4n^8} - \&c.$$
$$\&c.$$

$$l\,coſ.\frac{m\pi}{2n} = l(n-m) + l(n+m) - 2\,ln$$
$$- \frac{mm}{9nn} - \frac{m^4}{2.\,9^2n^4} - \frac{m^6}{3.\,9^3n^6} - \frac{m^8}{4.\,9^4n^8} - \&c.$$
$$- \frac{mm}{25nn} - \frac{m^4}{2.\,25^2n^4} - \frac{m^6}{3.\,25^3n^6} - \frac{m^8}{4.\,25^4n^8} - \&c.$$
$$- \frac{mm}{49nn} - \frac{m^4}{2.\,49^2n^4} - \frac{m^6}{3.\,49^3n^6} - \frac{m^8}{4.\,49^4n^8} - \&c.$$
$$\&c.$$

192. Occurrunt ergo in his Seriebus ſingulæ Poteſtates pares ipſius $\frac{m}{n}$, quæ ſunt multiplicatæ per Series, quarum ſummas jam ſupra aſſignavimus. Erit nempe

$$l\,ſin.\frac{m\pi}{2n} = lm + l(2n-m) + l(2n+m) - 3ln + l\pi - l8$$
$$- \frac{mm}{nn}\left(\frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} + \&c.\right)$$
$$- \frac{m^4}{2n^4}\left(\frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \frac{1}{10^4} + \frac{1}{12^4} + \&c.\right)$$
$$- \frac{m^6}{3n^6}\left(\frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \frac{1}{10^6} + \frac{1}{12^6} + \&c.\right)$$
$$- \frac{m^8}{4n^8}\left(\frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \frac{1}{10^8} + \frac{1}{12^8} + \&c.\right)$$
$$\&c.$$

l coſ.

$$l\,coſ.\frac{m\pi}{2n} = l(n-m) + l(n+m) - 2\,ln$$

$$-\frac{mm}{nn}\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.\right)$$
$$-\frac{m^4}{2n^4}\left(\frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.\right)$$
$$-\frac{m^6}{3n^6}\left(\frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.\right)$$
$$-\frac{m^8}{4n^8}\left(\frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \&c.\right)$$
&c.

Serierum poſteriorum modo ante (§. 190) ſummæ ſunt exhibitæ; priores Series quidem ex his derivari poſſent; at, quo facilius ad uſum tranferri queant, earum ſummas pariter hic adjiciam.

193. Quod ſi ergo, brevitatis gratia, ponamus

$$\alpha = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \&c.$$
$$\beta = \frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \&c.$$
$$\gamma = \frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \&c.$$
$$\delta = \frac{1}{2^8} + \frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \&c.$$
&c.

erunt ſummæ in numeris proxime expreſſæ hæ:

$$\alpha = 0,4112335167120566091181$$
$$\beta = 0,0676452021069461369697\,5$$
$$\gamma = 0,0158959853435070178080\,4$$
$$\delta = 0,0039221771726482200757\,0$$
$$\epsilon = 0,0009775337647732598489\,8$$
$$\zeta = 0,0002442007047249287227\,4$$

$$\eta = 0,\ 00006103889453949332915$$
$$\theta = 0,\ 00001525902225127269977$$
$$\iota = 0,\ 00000381471182744318008$$
$$\varkappa = 0,\ 00000095367522617534053$$
$$\lambda = 0,\ 00000023841863595259154$$
$$\mu = 0,\ 00000005960464832831555$$
$$\nu = 0,\ 00000001490116141589813$$
$$\xi = 0,\ 00000000372529031233986$$
$$o = 0,\ 00000000093132257548284$$
$$\pi = 0,\ 00000000023283064370807$$
$$\rho = 0,\ 00000000005820766091685$$
$$\sigma = 0,\ 00000000001455191522858$$
$$\tau = 0,\ 00000000000363797880710$$
$$\upsilon = 0,\ 00000000000090949470177$$
$$\varphi = 0,\ 00000000000022737367544$$
$$\chi = 0,\ 00000000000005684341886$$
$$\psi = 0,\ 00000000000001421085471$$
$$\omega = 0,\ 00000000000000355271367$$

reliquæ ſummæ in ratione quadrupla decreſcunt.

194. His ergo in ſubſidium vocatis, erit

$$l\,\text{ſin.}\,\frac{m\pi}{2n} = lm + l(2n - m) + l(2n + m) - 3\,ln + l\pi - l8$$
$$- \frac{mm}{nn}\left(\alpha - \frac{1}{2^2}\right) - \frac{m^4}{2n^4}\left(\beta - \frac{1}{2^4}\right) - \frac{m^6}{3n^6}\left(\gamma - \frac{1}{2^6}\right)$$
$$- \&c.$$

$$l\,\text{coſ.}\,\frac{m\pi}{2n} = l(n - m) + l(n + m) - 2\,ln$$
$$- \frac{mm}{nn}(A - 1) - \frac{m^4}{2n^4}(B - 1) - \frac{m^6}{3n^6}(C - 1) - \&c.,$$

quoniam igitur Logarithmi $l\pi$ & $l8$ dantur, erit

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Logarithmus hyperbolicus Sinus Anguli $\frac{m}{n}\, 90^{\circ} =$
$lm + l(2n - m) + l(2n + m) - 3ln$

$$- \frac{m^0}{n^0} \cdot 0,9347116558304357541\text{o}$$
$$- \frac{m^2}{n^2} \cdot 0,1612335167120566091\text{1}$$
$$- \frac{m^4}{n^4} \cdot 0,0025726010534730684\text{8}$$
$$- \frac{m^6}{n^6} \cdot 0,0000903284478356726\text{0}$$
$$- \frac{m^8}{n^8} \cdot 0,0000039817931620550\text{1}$$
$$- \frac{m^{10}}{n^{10}} \cdot 0,0000001942529546519\text{6}$$
$$- \frac{m^{12}}{n^{12}} \cdot 0,0000000100132874881\text{2}$$
$$- \frac{m^{14}}{n^{14}} \cdot 0,0000000005340413561\text{8}$$
$$- \frac{m^{16}}{n^{16}} \cdot 0,0000000000291485965\text{8}$$
$$- \frac{m^{18}}{n^{18}} \cdot 0,0000000000016179797\text{9}$$
$$- \frac{m^{20}}{n^{20}} \cdot 0,0000000000000909769\text{0}$$
$$- \frac{m^{22}}{n^{22}} \cdot 0,0000000000000051682\text{7}$$
$$- \frac{m^{24}}{n^{24}} \cdot 0,0000000000000002960\text{7}$$
$$- \frac{m^{26}}{n^{26}} \cdot 0,0000000000000000170\text{8}$$
$$- \frac{m^{28}}{n^{28}} \cdot 0,0000000000000000009\text{9}$$
$$- \frac{m^{30}}{n^{30}} \cdot 0,0000000000000000000\text{5}$$

LIB. I. At *Logarithmus hyperbolicus Coſinus Anguli* $\frac{m}{n}\,90^\circ =$

$$l(n - m) + l(n + m) - 2\,ln$$

$$- \frac{m^2}{n^2}.\ 0,\ 23370055013616982735$$
$$- \frac{m^4}{n^4}.\ 0,\ 00733901580209602727$$
$$- \frac{m^6}{n^6}.\ 0,\ 00048235888031404063$$
$$- \frac{m^8}{n^8}.\ 0,\ 00003879475632402982$$
$$- \frac{m^{10}}{n^{10}}.\ 0,\ 00000340827260896510$$
$$- \frac{m^{12}}{n^{12}}.\ 0,\ 00000031430809718659$$
$$- \frac{m^{14}}{n^{14}}.\ 0,\ 00000002989150274450$$
$$- \frac{m^{16}}{n^{16}}.\ 0,\ 0000000030464467239$$
$$- \frac{m^{18}}{n^{18}}.\ 0,\ 00000000028682639518$$
$$- \frac{m^{20}}{n^{20}}.\ 0,\ 000000000002868076974$$
$$- \frac{m^{22}}{n^{22}}.\ 0,\ 000000000000289697956$$
$$- \frac{m^{24}}{n^{24}}.\ 0,\ 0000000000000029506024$$
$$- \frac{m^{26}}{n^{26}}.\ 0,\ 00000000000000003026249$$
$$- \frac{m^{28}}{n^{28}}.\ 0,\ 000000000000000000312232$$
$$- \frac{m^{30}}{n^{30}}.\ 0,\ 0000000000000000000032379$$
$$- \frac{m^{32}}{n^{32}}.\ 0,\ 00000000000000000000003373$$
$$- \frac{m^{34}}{n^{34}}.\ 0,\ 000000000000000000000000352$$

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$$- \frac{m^{16}}{n^{16}} \cdot 0,00000000000000000037$$
$$- \frac{m^{18}}{n^{18}} \cdot 0,0000000000000000004$$

195. Si iſti Sinuum & Coſinuum Logarithmi hyperbolici multiplicentur per 0, 4342944819 &c., prodibant eorumdem Logarithmi vulgares ad Radium = 1 relati. Quoniam vero in Tabulis Logarithmus Sinus totius ſtatui ſolet = 10, quo Logarithmi tabulares Sinuum & Coſinuum obtineantur, poſt multiplicationem addi debet 10. Hinc erit

Logarithmus tabularis Sinus Anguli $\frac{m}{n} 90^\circ =$
$$lm + l(2n - m) + l(2n + m) - 3\,ln$$

$$+ \quad 9,594059885702190$$
$$- \frac{m^2}{n^2} \cdot 0,070022826605901$$
$$- \frac{m^4}{n^4} \cdot 0,001117266441661$$
$$- \frac{m^6}{n^6} \cdot 0,000039229146453$$
$$- \frac{m^8}{n^8} \cdot 0,000001729270798$$
$$- \frac{m^{10}}{n^{10}} \cdot 0,000000084362986$$
$$- \frac{m^{12}}{n^{12}} \cdot 0,000000004348715$$
$$- \frac{m^{14}}{n^{14}} \cdot 0,000000000231931$$
$$- \frac{m^{16}}{n^{16}} \cdot 0,000000000012659$$
$$- \frac{m^{18}}{n^{18}} \cdot 0,000000000000702$$
$$- \frac{m^{20}}{n^{20}} \cdot 0,000000000000039$$

LIB. I.

Logarithmus tabularis Coſinus Anguli $\frac{m}{n}\,90^\circ =$
$l(n-m)+l(n+m)-2\,ln$

$+$		10, 000000000000000
$-$	$\frac{m^2}{n^2}.$	0, 101494859341892
$-$	$\frac{m^4}{n^4}.$	0, 003187294065451
$-$	$\frac{m^6}{n^6}.$	0, 000209485800017
$-$	$\frac{m^8}{n^8}.$	0, 000016848348597
$-$	$\frac{m^{10}}{n^{10}}.$	0, 000001480193986
$-$	$\frac{m^{12}}{n^{12}}.$	0, 000000136502272
$-$	$\frac{m^{14}}{n^{14}}.$	0, 000000012981715
$-$	$\frac{m^{16}}{n^{16}}.$	0, 000000001261471
$-$	$\frac{m^{18}}{n^{18}}.$	0, 000000000124567
$-$	$\frac{m^{20}}{n^{20}}.$	0, 000000000012456
$-$	$\frac{m^{22}}{n^{22}}.$	0, 000000000001258
$-$	$\frac{m^{24}}{n^{24}}.$	0, 000000000000128
$-$	$\frac{m^{26}}{n^{26}}.$	0, 000000000000013

196. Harum ergo formularum ope inveniri poſſunt Logarithmi Sinuum & Coſinuum quorumvis Angulorum tam hyperbolici quam vulgares, etiam ignoratis ipſis Sinibus & Coſinibus. Ex Logarithmis autem Sinuum & Coſinuum per ſolam ſubtractionem inveniuntur Logarithmi Tangentium, Cotangentium, & Secantium, Coſecantiumque, quamobrem

pro his non erit opus peculiaribus formulis. Ceterum notandum eſt numerorum m, n, $n - m$, $n + m$, &c. Logarithmos hyperbolicos accipi oportere, cum Logarithmi hyperbolici Sinuum Coſinuumque quæruntur, vulgares autem, cum tales ope poſteriorum formularum ſunt indagandi. Præterea $m : n$ denotat rationem, quam Angulus propoſitus habet ad Angulum rectum; ſicque, cum Sinus Angulorum ſemirecto majorum æquentur Coſinibus Angulorum ſemirecto minorum ac viciſſim, fractio $\frac{m}{n}$ nunquam major accipienda erit quam $\frac{1}{2}$, hancque ob rem termini illi multo magis convergent, ut ſemiſſis inſtituto ſufficere poſſit. CAP. XI.

197. Antequam hoc argumentum relinquamus, aptiorem aperiamus modum Tangentes & Secantes quorumvis Angulorum inveniendi, quem Caput præcedens ſuppeditat. Quanquam enim Tangentes & Secantes per Sinus & Coſinus determinantur; tamen hoc fit per diviſionem, quæ operatio in tantis numeris nimis eſt operoſa. Ac Tangentes quidem & Cotangentes jam ſupra (§. 136) exhibuimus, verum illo loco rationem formularum reddere non licuit, quam huic Capiti reſervavimus.

198. Ex §. 181 ergo primum expreſſionem pro Tangente Anguli $\frac{m}{2n}\pi$ elicimus. Cum enim ſit $\frac{1}{nn - mm} + \frac{1}{9nn - mm} + \frac{1}{25nn - mm} + \&c. = \frac{\pi}{4mn}.\ tang.\ \frac{m}{2n}\pi$, erit $tang.\ \frac{m}{2n}\pi = \frac{4mn}{\pi}\left(\frac{1}{nn - mm} + \frac{1}{9nn - mm} + \frac{1}{25nn - mm} + \&c.\right)$. Cum deinde ſit $\frac{1}{nn - mm} + \frac{1}{4nn - mm} + \frac{1}{9nn - mm} + \&c. = \frac{1}{2mm} - \frac{\pi}{2mn}.\ cot.\ \frac{m}{n}\pi$, ſi pro n ſcribamus $2n$ erit $cot.\ \frac{m}{2n}\pi = \frac{2n}{m\pi} - \frac{4mn}{\pi}\left(\frac{1}{4nn - mm} + \frac{1}{16nn - mm} + \right.$

LIB. I. $\frac{1}{36nn - mm}$ + &c.). Convertantur hæ fractiones, præter primas, quippe quæ facile in computum ducuntur, in Series infinitas, erit

$$tang.\ \frac{m}{2n}\pi = \frac{mn}{nn - mm} \cdot \frac{4}{\pi}$$
$$+ \frac{4}{\pi}\left(\frac{m}{3^2 n} + \frac{m^3}{3^4 n^3} + \frac{m^5}{3^6 n^5} + \&c.\right)$$
$$+ \frac{4}{\pi}\left(\frac{m}{5^2 n} + \frac{m^3}{5^4 n^3} + \frac{m^5}{5^6 n^5} + \&c.\right)$$
$$+ \frac{4}{\pi}\left(\frac{m}{7^2 n} + \frac{m^3}{7^4 n^3} + \frac{m^5}{7^6 n^5} + \&c.\right)$$
$$\&c.$$

$$cot.\ \frac{m}{2n}\pi = \frac{n}{m} \cdot \frac{2}{\pi} - \frac{mn}{4nn - mm} \cdot \frac{4}{\pi}$$
$$- \frac{4}{\pi}\left(\frac{m}{4^2 n} + \frac{m^3}{4^4 n^3} + \frac{m^5}{4^6 n^5} + \&c.\right)$$
$$- \frac{4}{\pi}\left(\frac{m}{6^2 n} + \frac{m^3}{6^4 n^3} + \frac{m^5}{6^6 n^5} + \&c.\right)$$
$$- \frac{4}{\pi}\left(\frac{m}{8^2 n} + \frac{m^3}{8^4 n^3} + \frac{m^5}{8^6 n^5} + \&c.\right)$$
$$\&c.$$

198. At ex valore ipſius π cognito reperitur
$\frac{1}{\pi} = 0,\ 318309886183790671537767926745028724$,
deinde hic eædem Series occurrunt, quas ſupra litteris A, B, C, D, &c., & α, β, γ, δ, &c., indicavimus. His ergo notatis, erit

$$tang.\ \frac{m}{2n}\pi = \frac{mn}{nn - mm} \cdot \frac{4}{\pi} + \frac{m}{n} \cdot \frac{4}{\pi}(A - 1) + \frac{m^3}{n^3} \times \frac{4}{\pi}(B - 1) + \frac{m^5}{n^5} \cdot \frac{4}{\pi}(C - 1) + \frac{m^7}{n^7} \cdot \frac{4}{\pi}(D - 1)\ \&c.$$

Deinde erit pro Cotangente

cot,

$$cot.\ \frac{m}{2n}\pi = \frac{n}{m}\cdot\frac{2}{\pi} - \frac{4mn}{4nn-mm}\cdot\frac{1}{\pi} - \frac{m}{n}\cdot\frac{4}{\pi}\left(\alpha - \frac{1}{2^2}\right)$$
$$- \frac{m^3}{n^3}\cdot\frac{4}{\pi}\left(\beta - \frac{1}{2^4}\right) - \frac{m^5}{n^5}\cdot\frac{4}{\pi}\left(\gamma - \frac{1}{2^6}\right) - \&c.,$$

atque ex his formulis natæ ſunt expreſſiones, quas ſupra (§. 135) pro Tangente & Cotangente dedimus; ſimul vero (§. 137) oſtendimus, quomodo ex Tangentibus & Cotangentibus inventis per ſolam additionem & ſubtractionem Secantes & Coſecantes reperiantur. Harum ergo regularum ope univerſus Canon Sinuum, Tangentium & Secantium, eorumque Logarithmorum multo facilius ſupputari poſſet, quam quidem hoc a primis conditoribus eſt factum.

CAPUT XII.

De reali Functionum fractarum evolutione.

199. JAM ſupra, in Capite ſecundo, methodus eſt tradita Functionem quamcunque fractam in tot partes reſolvendi quot ejus denominator habeat Factores ſimplices; hi enim præbent denominatores fractionum illarum partialium. Ex quo manifeſtum eſt, ſi denominator quoſdam habeat Factores ſimplices imaginarios, fractiones quoque inde ortas fore imaginarias: his ergo caſibus parum juvabit fractionem realem in imaginarias reſolviſſe. Cum igitur oſtendiſſem omnem Functionem integram, qualis eſt denominator cujuſvis fractionis, quantumvis Factoribus ſimplicibus imaginariis ſcateat, tamen in factores duplices, ſeu ſecundæ dimenſionis, reales ſemper reſolvi poſſe; hoc modo in reſolutione fractionum quantitates imaginariæ evitari poterunt, ſi pro denominatoribus fractionum partialium non Factores denominatoris principalis ſimplices, ſed duplices reales aſſumamus.

200. Sit igitur propoſita hæc Functio fracta $\frac{M}{N}$, ex qua tot fractiones ſimplices ſecundum methodum ſupra expoſitam eliciantur, quot denominator N habuerit Factores ſimplices reales. Sit autem, loco imaginariorum, hæc expreſſio $pp - 2pqz.\,cof.\,\varphi + qqzz$ Factor ipſius N; &, quoniam in hoc negotio numeratorem & denominatorem in forma evoluta contemplari oportet, ſit hæc fractio propoſita

$$\frac{A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.}{(pp - 2pqz.\,cof.\,\varphi + qqzz)(\alpha + \beta z + \gamma zz + \delta z^3 + \&c.)},$$

ac ponatur fractio partialis ex denominatoris Factore $pp - 2pqz.\,cof.\,\varphi + qqzz$ oriunda hæc: $\frac{\mathrm{A} + \mathfrak{A}z}{pp - 2pqz.\,cof.\,\varphi + qqzz}$, quoniam enim variabilis z in denominatore duas habet dimenſiones, in numeratore unam habere poterit, non vero plures; alias enim integra Functio contineretur, quam ſeorſim elici oportet.

201. Sit, brevitatis gratia, numerator $A + Bz + Cz^2 + \&c. = M$ & alter denominatoris Factor $\alpha + \beta z + \gamma z^2 + \&c. = \mathrm{Z}$; ponatur altera pars ex denominatoris Factore Z oriunda $= \frac{Y}{\mathrm{Z}}$, eritque $Y = \frac{M - \mathrm{AZ} - \mathfrak{A}\mathrm{Z}z}{pp - 2pqz.\,cof.\,\varphi + qqzz}$, quæ expreſſio Functio integra ipſius z eſſe debet, ideoque neceſſe eſt ut $M - \mathrm{AZ} - \mathfrak{A}\mathrm{Z}z$, diviſibile ſit per $pp - 2pqz.\,cof.\,\varphi + qqzz$. Evaneſcet ergo $M - \mathrm{AZ} - \mathfrak{A}\mathrm{Z}z$, ſi ponatur $pp - 2pqz \times cof.\,\varphi + qqzz = 0$, hoc eſt ſi ponatur tam $z = \frac{p}{q}\,(cof.\,\varphi + \surd -1.\,fin.\,\varphi)$ quam $z = \frac{p}{q}\,(cof.\,\varphi - \surd -1.\,fin.\,\varphi)$; ſit $\frac{p}{q} = f$, eritque $z^n = f^n\,(cof.\,n\varphi \pm \surd -1.\,fin.\,n\varphi)$. Duplex ergo hic valor pro z ſubſtitutus duplicem dabit æquationem, unde ambas incognitas conſtantes A & $\mathfrak{A}$ definire licet.

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202. Facta ergo hac ſubſtitutione, æquatio $M = AZ + \mathfrak{A}Z\zeta$ evoluta hanc duplicem dabit æquationem C A P. XII.

$$\left.\begin{matrix} A + Bf.\,cof.\,\varphi + Cff.\,cof.\,2\varphi + Df^3.\,cof.\,3\varphi + \&c. \\ \pm (Bf.\,fin.\,\varphi + Cff.\,fin.\,2\varphi + Df^3.\,fin.\,3\varphi + \&c.)\sqrt{-1} \end{matrix}\right\} =$$

$$\left\{\begin{matrix} A(\alpha + \beta f.\,cof.\,\varphi + \gamma ff.\,cof.\,2\varphi + \delta f^3.\,cof.\,3\varphi + \&c.) \\ \pm A\,(\beta f.\,fin.\,\varphi + \gamma ff.\,fin.\,2\varphi + \delta f^3.\,fin.\,3\varphi + \&c.)\sqrt{-1} \\ \mp \mathfrak{A}\,(\alpha f.\,cof.\,\varphi + \beta ff.\,cof.\,2\varphi + \gamma f^3.\,cof.\,3\varphi + \&c.) \\ \pm \mathfrak{A}\,(\alpha f.\,fin.\,\varphi + \beta ff.\,fin.\,2\varphi + \gamma f^3.\,fin.\,3\varphi + \&c.)\sqrt{-1} \end{matrix}\right.$$

Sit, ad calculum abbreviandum,

$$\begin{aligned} A + Bf.\,cof.\,\varphi + Cff.\,cof.\,2\varphi + Df^3.\,cof.\,3\varphi + \&c. &= P \\ Bf.\,fin.\,\varphi + Cff.\,fin.\,2\varphi + Df^3.\,fin.\,3\varphi + \&c. &= \mathfrak{p} \\ \alpha + \beta f.\,cof.\,\varphi + \gamma ff.\,cof.\,2\varphi + \delta f^3.\,cof.\,3\varphi + \&c. &= Q \\ \beta f.\,fin.\,\varphi + \gamma ff.\,fin.\,2\varphi + \delta f^3.\,fin.\,3\varphi + \&c. &= \mathfrak{q} \\ \alpha f.\,cof.\,\varphi + \beta ff.\,cof.\,2\varphi + \gamma f^3.\,cof.\,3\varphi + \&c. &= R \\ \alpha f.\,fin.\,\varphi + \beta ff.\,fin.\,2\varphi + \gamma f^3.\,fin.\,3\varphi + \&c. &= \mathfrak{r} \end{aligned}$$

eritque, his poſitis,

$$P \pm \mathfrak{p}\sqrt{-1} = AQ \pm A\mathfrak{q}\sqrt{-1} + \mathfrak{A}R \pm \mathfrak{A}\mathfrak{r}\sqrt{-1}.$$

203. Ob ſignorum ambiguitatem hæ duæ oriuntur æquationes,

$$\begin{aligned} P &= AQ + \mathfrak{A}R \\ \mathfrak{p} &= A\mathfrak{q} + \mathfrak{A}\mathfrak{r} \end{aligned}$$

ex quibus incognitæ A & $\mathfrak{A}$ ita definiuntur, ut ſit

$$A = \frac{P\mathfrak{r} - \mathfrak{p}R}{Q\mathfrak{r} - \mathfrak{q}R}\ \&\ \mathfrak{A} = \frac{P\mathfrak{q} - \mathfrak{p}Q}{\mathfrak{q}R - Q\mathfrak{r}}.$$

Propoſita ergo fractione $\frac{M}{(pp - 2pq\zeta.\,cof.\,\varphi + qq\zeta\zeta)Z}$ per ſequentem regulam fractio partialis ex ea oriunda $\frac{A + \mathfrak{A}\zeta}{pp - 2pq\zeta.\,cof.\,\varphi + qq\zeta\zeta}$ definietur. Poſito $f = \frac{p}{q}$, & evo-

X 2

lutis ſingulis terminis, fiat ut ſequitur,

$$
\begin{array}{llll}
\text{poſito } z^n = f^n.\ coſ.\, n\,\varphi, & \text{fit} & M & = P \\
\ldots\ldots\ z^n = f^n.\ ſin.\, n\,\varphi, & \text{fit} & M & = \mathrm{p} \\
\ldots\ldots\ z^n = f^n.\ coſ.\, n\,\varphi, & \text{fit} & Z & = Q \\
\ldots\ldots\ z^n = f^n.\ ſin.\, n\,\varphi, & \text{fit} & Z & = \mathrm{q} \\
\ldots\ldots\ z^n = f^n.\ coſ.\, n\,\varphi, & \text{fit} & zZ & = R \\
\ldots\ldots\ z^n = f^n.\ ſin.\, n\,\varphi, & \text{fit} & zZ & = \mathrm{r}
\end{array}
$$

Inventis hoc modo valoribus P, Q, R, p, q, r erunt

$$A = \frac{P\mathrm{r} - \mathrm{p}R}{Q\mathrm{r} - \mathrm{q}R}; \quad \mathrm{a} = \frac{\mathrm{p}Q - P\mathrm{q}}{Q\mathrm{r} - \mathrm{q}R}.$$

EXEMPLUM I.

Si fuerit propoſita hæc Functio fracta $\frac{zz}{(1-z+zz)(1+z^4)}$ ex qua partem a denominatoris Factore $1-z+zz$ oriundam definire oporteat, quæ ſit $\frac{A+\mathrm{a}z}{1-z+zz}$. Ac primo quidem hic Factor, cum forma generali $pp - 2pqz.\ coſ.\,\varphi + qqzz$ comparatus, dat $p = 1$, $q = 1$ & $coſ.\,\varphi = \frac{1}{2}$, unde fit $\varphi = 60^o = \frac{\pi}{3}$. Quia itaque eſt $M = zz$, $Z = 1 + z^4$ & $f = 1$ erit

$$
\begin{array}{l}
P = \qquad coſ.\,\frac{2}{3}\pi = -\frac{1}{2}; \quad \mathrm{p} = \frac{\sqrt{3}}{2} \\
Q = 1 + coſ.\,\frac{4}{3}\pi = \frac{1}{2}; \quad \mathrm{q} = -\frac{\sqrt{3}}{2} \\
R = coſ.\,\frac{\pi}{3} + coſ.\,\frac{5\pi}{3} = 1; \quad \mathrm{r} = 0.
\end{array}
$$

Ex his invenitur $A = -1$ & $\mathrm{a} = 0$, ideoque fractio quæſita eſt $\frac{-1}{1-z+zz}$, hujuſque complementum erit

$\frac{1+z+zz}{1+z^4}$, cujus denominator $1+z^4$ cum habeat Factores $1+z\sqrt{2}+zz$ & $1-z\sqrt{2}+zz$, reſolutio denuo ſuſcipi poteſt; ſit autem $\varphi = \frac{\pi}{4}$ &, priori caſu $f = -1$, poſteriori $f = +1$. CAP. XII.

Exemplum II.

Sit igitur propoſita hæc fractio reſolvenda

$$\frac{1+z+zz}{(1+z\sqrt{2}+zz)(1-z\sqrt{2}+zz)}.$$

Erit $M = 1+z+zz$; & pro priore Factore habebitur $f = -1$, $\varphi = \frac{\pi}{4}$ & $Z = 1 - z\sqrt{2}+zz$, unde erit

$$\mathrm{P} = 1 - \quad cof.\frac{\pi}{4} + cof.\frac{2\pi}{4} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\mathrm{p} = \quad - \quad fin.\frac{\pi}{4} + fin.\frac{2\pi}{4} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\mathrm{Q} = 1 + \sqrt{2}.cof.\frac{\pi}{4} + cof.\frac{2\pi}{4} = 2$$

$$\mathrm{q} = \quad + \sqrt{2}.fin.\frac{\pi}{4} + fin.\frac{2\pi}{4} = 2$$

$$\mathrm{R} = - cof.\frac{\pi}{4} - \sqrt{2}.cof.\frac{2\pi}{4} - cof.\frac{3\pi}{4} = 0$$

$$\mathrm{r} = - fin.\frac{\pi}{4} - \sqrt{2}.fin.\frac{2\pi}{4} - fin.\frac{3\pi}{4} = -2\sqrt{2}$$

Ex his reperitur $\mathrm{Qr} - \mathrm{qR} = -4\sqrt{2}$: &
$\mathrm{A} = \frac{\sqrt{2}-1}{2\sqrt{2}}$, & $\mathrm{a} = 0$; unde ex denominatoris Factore $1+z\sqrt{2}+zz$ hæc orietur fractio partialis $\frac{(\sqrt{2}-1):2\sqrt{2}}{1+z\sqrt{2}+zz}$, alter autem Factor dabit ſimili modo hanc $\frac{(\sqrt{2}+1):2\sqrt{2}}{1-z\sqrt{2}+zz}$.
Hinc Functio primum propoſita $\frac{zz}{(1-z+zz)(1+z^4)}$ reſolvitur in has $\frac{-1}{1-z+zz} + \frac{(\sqrt{2}-1):2\sqrt{2}}{1+z\sqrt{2}+zz} + \frac{(\sqrt{2}+1):2\sqrt{2}}{1-z\sqrt{2}+zz}$.

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EXEMPLUM III.

Sit propoſita hæc fractio reſolvenda

$$\frac{1+2z+zz}{(1-\frac{8}{5}z+zz)(1+2z+3zz)}.$$

Pro Factore denominatoris $1-\frac{8}{5}z+zz$ oriatur iſta fractio $\frac{\mathrm{A}+\mathrm{a}z}{1-\frac{8}{5}z+zz}$; eritque $p=1$; $q=1$; $cof.\,\varphi=\frac{4}{5}$, unde $f=1$; $M=1+2z+zz$; $Z=1+2z+3zz$. Quia vero hic ratio Anguli φ ad rectum non conſtat, Sinus & Coſinus ejus multiplorum ſeorſim debent inveſtigari. Cum ſit

$$cof.\,\varphi=\frac{4}{5};\ \text{erit}\ fin.\,\varphi=\frac{3}{5}$$
$$cof.\,2\varphi=\frac{7}{25};\quad fin.\,2\varphi=\frac{24}{25}$$
$$cof.\,3\varphi=\frac{-44}{125};\quad fin.\,3\varphi=\frac{117}{125};$$

hinc fit

$$\mathrm{P}=1+2.\tfrac{4}{5}+\tfrac{7}{25}=\tfrac{72}{25}$$
$$\mathrm{p}=2.\tfrac{3}{5}+\tfrac{24}{25}=\tfrac{54}{25}$$
$$\mathrm{Q}=1+2.\tfrac{4}{5}+3.\tfrac{7}{25}=\tfrac{86}{25}$$
$$\mathrm{q}=2.\tfrac{3}{5}+3.\tfrac{24}{25}=\tfrac{102}{25}$$
$$\mathrm{R}=\tfrac{4}{5}+2.\tfrac{7}{25}-3.\tfrac{44}{125}=\tfrac{38}{125}$$
$$\mathrm{r}=\tfrac{3}{5}+2.\tfrac{24}{25}+3.\tfrac{117}{125}=\tfrac{666}{125}$$

ideoque $\mathrm{Qr}-\mathrm{qR}=\frac{53400}{25.125}=\frac{2136}{125}$. Ergo

$$\mathrm{A}=\frac{1836}{2136}=\frac{153}{178};\ \mathrm{a}=-\frac{540}{2136}=-\frac{45}{178}.$$

Quare fractio ex Factore $1 - \frac{8}{5}z + zz$ oriunda erit

$$\frac{9(17-5z):178}{1-\frac{8}{5}z+zz}.$$

Quæramus ſimili modo fractionem alteri Factori reſpondentem; erit $p = 1$, $q = -\sqrt{3}$ & *coſ.* $\varphi = \frac{1}{\sqrt{3}}$, ergo $f = -\frac{1}{\sqrt{3}}$, $M = 1 + 2z + zz$ & $Z = 1 - \frac{8}{5}z + zz$. Fiet autem, ob *coſ.* $\varphi = \frac{1}{\sqrt{3}}$, *ſin.* $\varphi = \frac{\sqrt{2}}{\sqrt{3}}$

coſ. $2\varphi = -\frac{1}{3}$, *ſin.* $2\varphi = \frac{2\sqrt{2}}{3}$

coſ. $3\varphi = -\frac{5}{3\sqrt{3}}$, *ſin.* $3\varphi = \frac{\sqrt{2}}{3\sqrt{3}}$

conſequenter

$$P = 1 - \frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{1}{3}\cdot - \frac{1}{3} = \frac{2}{9}$$

$$\text{p} = -\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3}\cdot\frac{2\sqrt{2}}{3} = -\frac{4\sqrt{2}}{9}$$

$$Q = 1 + \frac{8}{5\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{1}{3}\cdot - \frac{1}{3} = \frac{64}{45}$$

$$\text{q} = +\frac{8}{5\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3}\cdot\frac{2\sqrt{2}}{3} = \frac{34\sqrt{2}}{45}$$

$$R = -\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} - \frac{8}{5\cdot 3}\cdot - \frac{1}{3} - \frac{1}{3\sqrt{3}}\cdot - \frac{5}{3\sqrt{3}} = \frac{4}{135}$$

$$\text{r} = -\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} - \frac{8}{5\cdot 3}\cdot\frac{2\sqrt{2}}{3} - \frac{1}{3\sqrt{3}}\cdot\frac{\sqrt{2}}{3\sqrt{3}} = -\frac{98\sqrt{2}}{135}$$

ideoque $Q\,\text{r} = \text{q}\,R = -\frac{712\sqrt{2}}{675}$; fiet ergo

$$A = \frac{100}{712} = \frac{25}{178};\quad \text{a} = \frac{540}{712} = \frac{135}{178}.$$

Fractio ergo propoſita $\frac{1+2z+zz}{(1-\frac{8}{5}z+zz)(1+2z+3zz)}$ reſolvitur in $\frac{9(17-5z):178}{1-\frac{8}{5}z+zz} + \frac{5(5+27z):178}{1+2z+3zz}$.

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204. Poſſunt autem valores litterarum R & r ex litteris Q & q definiri; cum enim ſit

$$Q = \alpha + \beta f.\,cof.\,\varphi + \gamma f^2.\,cof.\,2\varphi + \delta f^3.\,cof.\,3\varphi \;\&c.$$
$$q = \beta f.\,fin.\,\varphi + \gamma f^2.\,fin.\,2\varphi + \delta f^3.\,fin.\,3\varphi \;\&c.$$

erit

$$Q.\,cof.\,\varphi - q.\,fin.\,\varphi = \alpha.\,cof.\,\varphi + \beta f.\,cof.\,2\varphi + \gamma f^2.\,cof.\,3\varphi + \&c.$$

ideoque $R = f\,(Q.\,cof.\,\varphi - q.\,fin.\,\varphi)$

deinde erit

$$Q.\,fin.\,\varphi + q.\,cof.\,\varphi = \alpha.\,fin.\,\varphi + \beta f.\,fin.\,2\varphi + \gamma ff.\,fin.\,3\varphi + \&c.$$

ergo $r = f.\,(Q.\,fin.\,\varphi + q.\,cof.\,\varphi)$.

Ex his porro fit

$$Qr - qR = (QQ + qq)\,f.\,fin.\,\varphi$$
$$Pr - pR = (PQ + pq)\,f.\,fin.\,\varphi + (Pq - pQ)\,f.\,cof.\,\varphi$$

eritque conſequenter

$$A = \frac{PQ + pq}{QQ + qq} + \frac{Pq - pQ}{QQ + qq}.\,\frac{cof.\,\varphi}{fin.\,\varphi}$$
$$a = -\frac{Pq + pQ}{(QQ + qq)\,f.\,fin.\,\varphi}.$$

Quare ex denominatoris Factore $pp - 2pqz.\,cof.\,\varphi + qqzz$ naſcitur iſta fractio partialis

$$\frac{(PQ + pq)\,f.\,fin.\,\varphi + (Pq - pQ)\,(f.\,cof.\,\varphi - z)}{(pp - 2pqz.\,cof.\,\varphi + qqzz)\,(QQ + qq)\,f.\,fin.\,\varphi}$$

ſeu, ob $f = \frac{p}{q}$, hæc

$$\frac{(PQ + pq)\,p.\,fin.\,\varphi + (Pq - pQ)\,(p.\,cof.\,\varphi - qz)}{(pp - 2pqz.\,cof.\,\varphi + qqzz)\,(QQ + qq)\,p.\,fin.\,\varphi}.$$

205. Oritur ergo hæc fractio partialis ex Functionis propoſitæ $\frac{M}{(pp - 2pqz.\,cof.\,\varphi + qqzz)\,Z}$ Factore denominatoris $pp - 2pqz.\,cof.\,\varphi + qqzz$, atque litteræ P, p, Q & q ſequenti modo ex Functionibus M & Z inveniuntur:

poſito

poſito $z^n = \frac{p^n}{q^n}.\ coſ.\ n\varphi$, fit $M = \mathrm{P}$,

& $Z = \mathrm{Q}$;

& poſito $z^n = \frac{p^n}{q^n}.\ ſin.\ n\varphi$, fit $M = \mathfrak{p}$,

& $Z = \mathfrak{q}$:

ubi notandum eſt Functiones M & Z, antequam hæc ſubſtitutio fiat, omnino evolvi debere, ut hujuſmodi habeant formas

$$M = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.,$$
$$\& \ Z = \alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.;$$

eritque ideo

$$\mathrm{P} = A + B\frac{p}{q}.\ coſ.\varphi + C\frac{p^2}{q^2}.\ coſ.\ 2\varphi + D\frac{p^3}{q^3}.\ coſ.\ 3\varphi + \&c.$$
$$\mathfrak{p} = \quad B\frac{p}{q}.\ ſin.\varphi + C\frac{p^2}{q^2}.\ ſin.\ 2\varphi + D\frac{p^3}{q^3}.\ ſin.\ 3\varphi + \&c.$$
$$\mathrm{Q} = \alpha + \beta\frac{p}{q}.\ coſ.\varphi + \gamma\frac{p^2}{q^2}.\ coſ.\ 2\varphi + \delta\frac{p^3}{q^3}.\ coſ.\ 3\varphi + \&c.$$
$$\mathfrak{q} = \quad \beta\frac{p}{q}.\ ſin.\varphi + \gamma\frac{p^2}{q^2}.\ ſin.\ 2\varphi + \delta\frac{p^3}{q^3}.\ ſin.\ 3\varphi + \&c..$$

206. Ex præcedentibus autem intelligitur hanc reſolutionem locum habere non poſſe, ſi functio Z eundem Factorem $pp - 2pqz.\ coſ.\varphi + qqzz$ adhuc in ſe complectatur; hoc enim caſu in æquatione $M = \mathrm{A}Z + \mathfrak{a}Zz$ facta ſubſtitutione $z^n = f^n\ (coſ.\ n\varphi \pm \surd -1.\ ſin.\ n\varphi)$, ipſa quantitas Z evaneſceret, nihilque propterea colligi poſſet. Quamobrem, ſi Functionis fractæ $\frac{M}{N}$ denominator habeat Factorem $(pp - 2pqz.\ coſ.\varphi + qqzz)^2$ vel altiorem Poteſtatem, peculiari opus erit reſolutione. Sit igitur $N = (pp - 2pqz.\ coſ.\varphi + qqzz)^2 Z$; atque ex denominatoris Factore $(pp - 2pqz.\ coſ.\varphi + qqzz)^2$ orientur hujuſmodi duæ fractiones partiales

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 $\frac{A + \mathfrak{A}z}{(pp - 2pqz.\ cof.\ \varphi + qqzz)^2} + \frac{B + \mathfrak{B}z}{pp - 2pqz.\ cof.\ \varphi + qqzz}$, ubi litteras conſtantes A, $\mathfrak{A}$, B, $\mathfrak{B}$ determinari oportet.

207. His poſitis, debebit iſta expreſſio

$$\frac{M - (A + \mathfrak{A}z)\,Z - (B + \mathfrak{B}z)\,Z\,(pp - 2pqz.\ cof.\ \varphi + qqzz)}{(pp - 2pqz.\ cof.\ \varphi + qqzz)^2}$$

eſſe Functio integra, & hanc ob rem numerator diviſibilis erit per denominatorem. Primum ergo hæc expreſſio $M - AZ - \mathfrak{A}Zz$ diviſibilis eſſe debet per $pp - 2pqz.\ cof.\ \varphi + qqzz$; qui cum ſit caſus præcedens, eodem quoque modo litteræ A & $\mathfrak{A}$ determinabuntur.

Quare, poſito $z^n = \frac{p^n}{q^n}.\ cof.\ n\varphi$, ſit $M = P$,
& $z = N$:

&, poſito $z^n = \frac{p^n}{q^n}.\ ſin.\ n\varphi$, ſit $M = \mathfrak{P}$,
& $z = \mathfrak{N}$.

Hiſque factis ſecundum regulam ſupra datam, erit

$$A = \frac{PN + \mathfrak{P}\mathfrak{N}}{N^2 + \mathfrak{N}^2} + \frac{P\mathfrak{N} - \mathfrak{P}N}{N^2 + \mathfrak{N}^2} \cdot \frac{cof.\ \varphi}{ſin.\ \varphi}$$

$$\mathfrak{A} = - \frac{P\mathfrak{N} + \mathfrak{P}N}{N^2 + \mathfrak{N}^2} \cdot \frac{q}{p.\ ſin.\ \varphi}.$$

208. Inventis ergo hoc modo A & $\mathfrak{A}$, fiet $\frac{M - (A + \mathfrak{A}z)\,Z}{pp - 2pqz.\ cof.\ \varphi + qqzz}$ Functio integra, quæ ſit $= P$; atque ſupereſt ut $P - BZ - \mathfrak{B}Zz$ diviſibile evadat per $pp - 2pqz.\ cof.\ \varphi + qqzz$, quæ expreſſio cum ſimilis ſit præcedenti, ſi

poſito $z^n = \frac{p^n}{q^n}.\ cof.\ n\varphi$, vocetur $P = R$,

&, poſito $z^n = \frac{p^n}{q^n}.\ ſin.\ n\varphi$, vocetur $P = \mathfrak{R}$; erit

$$B = \frac{\mathrm{R}N + \mathrm{r}n}{N^2 + n^2} + \frac{\mathrm{R}n - \mathrm{r}N}{N^2 + n^2} \cdot \frac{cof.\,\varphi}{ſin.\,\varphi}$$

$$b = -\frac{\mathrm{R}n + \mathrm{r}N}{N^2 + n^2} \cdot \frac{q}{p.\,ſin.\,\varphi}.$$

209. Hinc jam generaliter concludere licet quomodo reſolutio inſtitui debeat, ſi denominator Functionis propoſitæ $\frac{M}{N}$, Factorem habeat $(pp - 2pqz.\,cof.\,\varphi + qqzz)^k$: ſit enim $N = (pp - 2pqz.\,cof.\,\varphi + qqzz)^k Z$, ita ut hæc reſolvenda ſit Functio fracta

$$\frac{M}{(pp - 2pqz.\,cof.\,\varphi + qqzz)^k Z}.$$

Præbeat ergo Factor denominatoris $(pp - 2pqz.\,cof.\,\varphi + qqzz)^k$ has partes:

$$\frac{\mathrm{A} + \mathfrak{a}z}{(pp - 2pqz.\,cof.\,\varphi + qqzz)^k} + \frac{\mathrm{B} + \mathfrak{b}z}{(pp - 2pqz.\,cof.\,\varphi + qqzz)^{k-1}} +$$
$$\frac{\mathrm{C} + \mathfrak{c}z}{(pp - 2pqz.\,cof.\,\varphi + qqzz)^{k-2}} + \frac{\mathrm{D} + \mathfrak{d}z}{(pp - 2pqz.\,cof.\,\varphi + qqzz)^{k-3}} + \&c.$$

Jam, poſito $z^n = \frac{p^n}{q^n}.\,cof.\,n\varphi$, ſit $M = \mathrm{M}$,
& $z = \mathrm{N}$;

&, poſito $z^n = \frac{p^n}{q^n}.\,ſin.\,n\varphi$, ſit $M = \mathrm{m}$,
& $z = \mathrm{n}$;

erit

$$\mathrm{A} = \frac{\mathrm{MN} + \mathrm{mn}}{\mathrm{N}^2 + \mathrm{n}^2} + \frac{\mathrm{Mn} - \mathrm{mN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{cof.\,\varphi}{ſin.\,\varphi}$$

$$\mathfrak{a} = -\frac{\mathrm{Mn} + \mathrm{mN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{q}{p.\,ſin.\,\varphi}.$$

Deinde vocetur $\frac{M - (\mathrm{A} + \mathfrak{a}z)Z}{pp - 2pqz.\,cof.\,\varphi + qqzz} = P$; atque,

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poſito $z^n = \frac{p^n}{q^n} \cdot \text{coſ.}\, n\varphi$, ſit $P = \mathrm{P}$,

&, poſito $z^n = \frac{p^n}{q^n} \cdot \text{ſin.}\, n\varphi$, ſit $P = \mathrm{p}$;

erit

$$\mathrm{B} = \frac{\mathrm{PN} + \mathrm{pn}}{\mathrm{N}^2 + \mathrm{n}^2} + \frac{\mathrm{Pn} - \mathrm{pN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{\text{coſ.}\, \varphi}{\text{ſin.}\, \varphi}$$

$$\mathrm{b} = -\frac{\mathrm{Pn} + \mathrm{pN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{q}{p.\, \text{ſin.}\, \varphi}.$$

Tum vocetur $\frac{P - (\mathrm{B} + \mathrm{b}z)\, Z}{pp - 2pqz.\, \text{coſ.}\, \varphi + qqzz} = Q$, atque

poſito $z^n = \frac{p^n}{q^n} \cdot \text{coſ.}\, n\varphi$, ſit $Q = \mathrm{Q}$,

&, poſito $z^n = \frac{p^n}{q^n} \cdot \text{ſin.}\, n\varphi$, ſit $Q = \mathrm{q}$;

erit

$$\mathrm{C} = \frac{\mathrm{QN} + \mathrm{qn}}{\mathrm{N}^2 + \mathrm{n}^2} + \frac{\mathrm{Qn} - \mathrm{qN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{\text{coſ.}\, \varphi}{\text{ſin.}\, \varphi}$$

$$\mathrm{c} = -\frac{\mathrm{Qn} + \mathrm{qN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{q}{p.\, \text{ſin.}\, \varphi}.$$

Porro vocetur $\frac{Q - (\mathrm{C} + \mathrm{c}z)\, Z}{pp - 2pqz.\, \text{coſ.}\, \varphi + qqzz} = R$, atque

poſito $z^n = \frac{p^n}{q^n} \cdot \text{coſ.}\, n\varphi$, ſit $R = \mathrm{R}$,

&, poſito $z^n = \frac{p^n}{q^n} \cdot \text{ſin.}\, n\varphi$, ſit $R = \mathrm{r}$;

erit

$$\mathrm{D} = \frac{\mathrm{RN} + \mathrm{rn}}{\mathrm{N}^2 + \mathrm{n}^2} + \frac{\mathrm{Rn} - \mathrm{rN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{\text{coſ.}\, \varphi}{\text{ſin.}\, \varphi}$$

$$\mathrm{d} = -\frac{\mathrm{Rn} + \mathrm{rN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{q}{p.\, \text{ſin.}\, \varphi}.$$

Hocque modo progrediendum eſt donec ultimæ fractionis, cujus denominator eſt $pp - 2pqz.\ cof.\ \varphi + qqzz$, numerator fuerit determinatus. CAP. XII.

E X E M P L U M.

Sit iſta propoſita Functio fracta

$$\frac{z - z^3}{(1 + zz)^4\,(1 + z^4)}$$

ex cujus denominatoris Factore $(1 + zz)^4$ oriantur hæ fractiones partiales,

$$\frac{A + \mathrm{a}z}{(1 + zz)^4} + \frac{B + \mathrm{b}z}{(1 + zz)^3} + \frac{C + \mathrm{c}z}{(1 + zz)^2} + \frac{D + \mathrm{d}z}{1 + zz}.$$

Comparatione ergo inſtituta, erit $p = 1$, $q = 1$, $cof.\ \varphi = 0$; ideoque $\varphi = \frac{1}{2}\pi$, porroque $M = z - z^3$ & $Z = 1 + z^4$. Hinc erit $M = 0$; $\mathrm{m} = 2$; $N = 2$; $\mathrm{n} = 0$, & $ſin.\ \varphi = 1$.

Hinc itaque invenitur

$$A = -\frac{4}{4}.\ 0 = 0, \ \&\ \mathrm{a} = 1.$$

Ergo $A + \mathrm{a}z = z$; hincque $P = \frac{z - z^3 - z - z^3}{1 + zz} = -z^3$,

& $P = 0$, $\mathrm{p} = 1$: unde reperitur

$$B = 0, \ \&\ \mathrm{b} = \frac{1}{2}.$$

Ergo $B + \mathrm{b}z = \frac{1}{2}z$, & $Q = \frac{-z^3 - \frac{1}{2}z - \frac{1}{2}z^3}{1 + zz} = -\frac{1}{2}z - \frac{1}{2}z^3$;

unde $Q = 0$ & $\mathrm{q} = 0$, ergo

$C = 0$ & $\mathrm{c} = 0$, hincque $R = -\frac{\frac{1}{2}z - \frac{1}{2}z^3}{1 + zz} = -\frac{1}{2}z$,

ergo $R = 0$; $\mathrm{r} = -\frac{1}{2}$; unde fit

$$D = 0 \ \&\ \mathrm{d} = -\frac{1}{4}.$$

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LIB. I. Quamobrem fractiones quæſitæ ſunt hæ

$\frac{z}{(1+zz)^3} + \frac{z}{2(1+zz)^2} - \frac{z}{4(1+zz)}$. Reliquæ vero fractionis numerator eſt $= S = \frac{R-(D+\mathfrak{d}z)Z}{1+zz} = -\frac{1}{4}z + \frac{1}{4}z^3$, quæ ergo erit $= \frac{-z+z^3}{4(1+z^4)}$.

210. Hac ergo methodo ſimul innoteſcit fractio complementi, quæ cum inventis conjuncta producat fractionem propoſitam ipſam. Scilicet ſi fractionis

$$\frac{M}{(pp - 2pqz.\,coſ.\,\varphi + qqzz)^k\,Z}$$

inventæ fuerint omnes fractiones partiales ex Factore $(pp - 2pqz.\,coſ.\,\varphi + qqzz)^k$ oriundæ, pro quibus formati ſunt valores Functionum P, Q, R, S, T, ſi harum litterarum Series ulterius continuetur, erit ea, quæ ultimam, qua opus eſt ad numeratores inveniendos, ſequitur, numerator reliquæ fractionis denominatorem Z habentis; nempe, ſi $k = 1$, erit reliqua fractio $\frac{P}{Z}$; ſi $k = 2$, erit reliqua fractio $\frac{Q}{Z}$; ſi $k = 3$, erit ea $\frac{R}{Z}$, & ita porro. Inventa autem hac reliqua fractione denominatorem Z habente, ea per has regulas ulterius reſolvi poterit.

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CAPUT XIII.

De Seriebus recurrentibus.

211. Ad hoc Serierum genus, quas Moivræus *recurrentes* vocare ſolet, hic refero omnes Series quæ ex evolutione Functionis cujuſque fractæ per diviſionem actualem inſtituta naſcuntur. Supra enim jam oſtendimus has Series ita eſſe comparatas, ut quivis terminus ex aliquot præcedentibus ſecundum legem quandam conſtantem determinetur, quæ lex a denominatore Functionis fractæ pendet. Cum autem nunc Functionem quamcunque fractam in alias ſimpliciores reſolvere docuerim, hinc Series quoque recurrens in alias ſimpliciores reſolvetur. In hoc igitur Capite propoſitum eſt Serierum recurrentium cujuſvis gradus reſolutionem in ſimpliciores exponere.

212. Sit propoſita iſta Functio fracta genuina

$$\frac{a + bz + czz + dz^3 + \&c.}{1 - \alpha z - \beta zz - \gamma z^3 - \delta z^4 - \&c.}$$

quæ per diviſionem evolvatur in hanc Seriem recurrentem

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus coëfficientes quemadmodum progrediantur, ſupra eſt oſtenſum. Quod ſi jam Functio illa fracta reſolvatur in fractiones ſuas ſimplices, & unaquæque in Seriem recurrentem evolvatur, manifeſtum eſt ſummam omnium harum Serierum ex fractionibus partialibus ortarum æqualem eſſe debere Seriei recurrenti

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.$$

Fractiones ergo partiales, quas ſupra invenire docuimus, dabunt

LIB. I. expreſſio illi eſt æqualis, id quod eliminatis denominatoribus patebit, fiet enim,

$$1.2.3\ldots\ldots n(n+1)\ldots\ldots(n+k-1) = 1.2.3\ldots\ldots(k-1)k\ldots\ldots(k+n-1)$$

quæ eſt æquatio identica.

216. Quoties ergo in reſolutione Functionum fractarum ad hujuſmodi fractiones partiales $\frac{A}{(1-pz)^k}$ pervenitur, toties Seriei recurrentis ex illa Functione fracta ortæ $A + Bz + Cz^2 + Dz^3 +$ &c., terminus generalis aſſignari poterit, quippe qui erit ſumma terminorum generalium Serierum, quæ ex fractionibus partialibus naſcuntur.

EXEMPLUM I.

Invenire terminum generalem Seriei recurrentis, quæ ex hac fractione $\frac{1-z}{1-z-2zz}$ *naſcitur.*

Series hinc nata eſt $1 + 0z + 2zz + 2z^3 + 6z^4 + 10z^5 + 22z^6 + 42z^7 + 86z^8 +$ &c. Ad coëfficientem poteſtatis generalis z^n inveniendum, fractio $\frac{1-z}{1-z-2zz}$ reſolvatur in $\frac{\frac{2}{3}}{1+z} + \frac{\frac{1}{3}}{1-2z}$, unde oritur terminus generalis quæſitus $(\frac{2}{3}(-1)^n + \frac{1}{3}\cdot 2^n)z^n = \frac{2^n \pm 2}{3}z^n$, ubi ſignum + valet ſi n ſit numerus par, ſignum — ſi n ſit impar.

EXEMPLUM II.

Invenire terminum generalem Seriei recurrentis quæ oritur ex fractione $\frac{1-z}{1-5z+6zz}$, *ſeu Seriei hujus* $1 + 4z + 14z^2 + 46z^3 + 146z^4 + 454z^5 +$ &c.

Ob denominatorem $= (1 - 2z)(1 - 3z)$ reſolvitur fractio in has $\frac{-1}{1 - 2z} + \frac{2}{1 - 3z}$, ex quibus fit terminus generalis $2.\, 3^n z^n - 2^n z^n = (2.\, 3^n - 2^n) z^n$.

EXEMPLUM III.

Invenire terminum generalem Seriei hujus $1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + 29z^6 + 47z^7 +$ &c., *quæ oritur ex evolutione fractionis* $\frac{1 + 2z}{1 - z - zz}$.

Ob denominatoris Factores $1 - \left(\frac{1 + \sqrt{5}}{2}\right) z$ & $1 - \left(\frac{1 - \sqrt{5}}{2}\right) z$, per reſolutionem prodeunt $\frac{\frac{1 + \sqrt{5}}{2}}{1 - \left(\frac{1 + \sqrt{5}}{2}\right) z} + \frac{\frac{1 - \sqrt{5}}{2}}{1 - \left(\frac{1 - \sqrt{5}}{2}\right) z}$, unde erit terminus generalis $= \left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} z^n + \left(\frac{1 - \sqrt{5}}{2}\right)^{n+1} z^n$.

EXEMPLUM IV.

Invenire terminum generalem Seriei hujus
$a + (\alpha a + b) z + (\alpha^2 a + \alpha b + \beta a) z^2 + (\alpha^3 a + \alpha^2 b + 2\alpha\beta a + \beta b) z^3 +$ &c., *quæ oritur ex evolutione fractionis* $\frac{a + bz}{1 - \alpha z - \beta zz}$.

Per reſolutionem oriuntur hæ duæ fractiones :

204. Poſſunt autem valores litterarum R & r ex litteris Q & q definiri; cum enim ſit

$$Q = \alpha + \beta f. cof. \varphi + \gamma f^2. cof. 2\varphi + \delta f^3. cof. 3\varphi \text{ \&c.}$$
$$\mathrm{q} = \qquad \beta f. fin. \varphi + \gamma f^2. fin. 2\varphi + \delta f^3. fin. 3\varphi \text{ \&c.}$$

erit

$$Q. cof. \varphi - \mathrm{q}. fin. \varphi = \alpha. cof. \varphi + \beta f. cof. 2\varphi + \gamma f^2. cof. 3\varphi + \text{\&c.}$$

ideoque $R = f\,(Q.\, cof.\, \varphi - \mathrm{q}.\, fin.\, \varphi)$

deinde erit

$$Q. fin. \varphi + \mathrm{q}. cof. \varphi = \alpha. fin. \varphi + \beta f. fin. 2\varphi + \gamma ff. fin. 3\varphi + \text{\&c.}$$

ergo $\mathrm{r} = f.\,(Q.\, fin.\, \varphi + \mathrm{q}.\, cof.\, \varphi)$.

Ex his porro fit

$$Q\mathrm{r} - \mathrm{q}R = (QQ + \mathrm{qq})\, f.\, fin.\, \varphi$$
$$P\mathrm{r} - \mathrm{p}R = (PQ + \mathrm{pq})\, f.\, fin.\, \varphi + (P\mathrm{q} - \mathrm{p}Q)\, f.\, cof.\, \varphi$$

eritque conſequenter

$$A = \frac{PQ + \mathrm{pq}}{QQ + \mathrm{qq}} + \frac{P\mathrm{q} - \mathrm{p}Q}{QQ + \mathrm{qq}} \cdot \frac{cof.\, \varphi}{fin.\, \varphi}$$
$$\mathrm{a} = - \frac{P\mathrm{q} + \mathrm{p}Q}{(QQ + \mathrm{qq})\, f.\, fin.\, \varphi}.$$

Quare ex denominatoris Factore $pp - 2pqz.\ cof.\ \varphi + qqzz$ naſcitur iſta fractio partialis

$$\frac{(PQ + \mathrm{pq})\, f.\, fin.\, \varphi + (P\mathrm{q} - \mathrm{p}Q)\,(f.\, cof.\, \varphi - z)}{(pp - 2pqz.\, cof.\, \varphi + qqzz)\,(QQ + \mathrm{qq})\, f.\, fin.\, \varphi}$$

ſeu, ob $f = \frac{p}{q}$, hæc

$$\frac{(PQ + \mathrm{pq})\, p.\, fin.\, \varphi + (P\mathrm{q} - \mathrm{p}Q)\,(p.\, cof.\, \varphi - qz)}{(pp - 2pqz.\, cof.\, \varphi + qqzz)\,(QQ + \mathrm{qq})\, p.\, fin.\, \varphi}.$$

205. Oritur ergo hæc fractio partialis ex Functionis propoſitæ $\frac{M}{(pp - 2pqz.\ cof.\ \varphi + qqzz)\,Z}$ Factore denominatoris $pp - 2pqz.\ cof.\ \varphi + qqzz$, atque litteræ P, p, Q & q ſequenti modo ex Functionibus M & Z inveniuntur:

poſito

poſito $z^n = \frac{p^n}{q^n}. \, cof. \, n\varphi$, fit $M = \mathrm{P}$,
& $Z = \mathrm{Q}$;

& poſito $z^n = \frac{p^n}{q^n}. \, ſin. \, n\varphi$, fit $M = \mathfrak{p}$,
& $Z = \mathfrak{q}$:

ubi notandum eſt Functiones M & Z, antequam hæc ſubſtitutio fiat, omnino evolvi debere, ut hujuſmodi habeant formas

$$M = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.,$$
$$\& \; Z = \alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \&c.;$$

eritque ideo

$$\mathrm{P} = A + B\tfrac{p}{q}. \, cof. \, \varphi + C\tfrac{p^2}{q^2}. \, cof. \, 2\varphi + D\tfrac{p^3}{q^3}. \, cof. \, 3\varphi + \&c.$$
$$\mathfrak{p} = \qquad B\tfrac{p}{q}. \, ſin. \, \varphi + C\tfrac{p^2}{q^2}. \, ſin. \, 2\varphi + D\tfrac{p^3}{q^3}. \, ſin. \, 3\varphi + \&c.$$
$$\mathrm{Q} = \alpha + \beta\tfrac{p}{q}. \, cof. \, \varphi + \gamma\tfrac{p^2}{q^2}. \, cof. \, 2\varphi + \delta\tfrac{p^3}{q^3}. \, cof. \, 3\varphi + \&c.$$
$$\mathfrak{q} = \qquad \beta\tfrac{p}{q}. \, ſin. \, \varphi + \gamma\tfrac{p^2}{q^2}. \, ſin. \, 2\varphi + \delta\tfrac{p^3}{q^3}. \, ſin. \, 3\varphi + \&c..$$

206. Ex præcedentibus autem intelligitur hanc reſolutionem locum habere non poſſe, ſi functio Z eundem Factorem $pp - 2pqz. \, cof. \, \varphi + qqzz$ adhuc in ſe complectatur; hoc enim caſu in æquatione $M = \mathfrak{A}Z + \mathfrak{a}Zz$ facta ſubſtitutione $z^n = f^n\,(cof. \, n\varphi \pm \sqrt{-1}. \, ſin. \, n\varphi)$, ipſa quantitas Z evaneſceret, nihilque propterea colligi poſſet. Quamobrem, ſi Functionis fractæ $\frac{M}{N}$ denominator habeat Factorem $(pp - 2pqz. \, cof. \, \varphi + qqzz)^2$ vel altiorem Poteſtatem, peculiari opus erit reſolutione. Sit igitur $N = (pp - 2pqz. \, cof. \, \varphi + qqzz)^2 Z$; atque ex denominatoris Factore $(pp - 2pqz. \, cof. \, \varphi + qqzz)^2$ orientur hujuſmodi duæ fractiones partiales

 $\frac{\mathrm{A} + \text{\textsc{a}}z}{(pp - 2pqz.\, \cos.\varphi + qqzz)^2} + \frac{\mathrm{B} + \text{\textsc{b}}z}{pp - 2pqz.\, \cos.\varphi + qqzz}$, ubi litteras conſtantes A, $\text{\textsc{a}}$, B, $\text{\textsc{b}}$ determinari oportet.

207. His poſitis, debebit iſta expreſſio

$$\frac{M - (\mathrm{A} + \text{\textsc{a}}z)Z - (\mathrm{B} + \text{\textsc{b}}z)Z(pp - 2pqz.\, \cos.\varphi + qqzz)}{(pp - 2pqz.\, \cos.\varphi + qqzz)^2}$$

eſſe Functio integra, & hanc ob rem numerator diviſibilis erit per denominatorem. Primum ergo hæc expreſſio $M - \mathrm{A}Z - \text{\textsc{a}}Zz$ diviſibilis eſſe debet per $pp - 2pqz.\, \cos.\varphi + qqzz$; qui cum ſit caſus præcedens, eodem quoque modo litteræ A & $\text{\textsc{a}}$ determinabuntur.

Quare, poſito $z^n = \frac{p^n}{q^n}.\, \cos.\, n\varphi$, ſit $M = \mathrm{P}$,

& $z = \mathrm{N}$:

&, poſito $z^n = \frac{p^n}{q^n}.\, \sin.\, n\varphi$, ſit $M = \text{\textsc{p}}$,

& $z = \text{\textsc{n}}$.

Hiſque factis ſecundum regulam ſupra datam, erit

$$\mathrm{A} = \frac{\mathrm{PN} + \text{\textsc{pn}}}{\mathrm{N}^2 + \text{\textsc{n}}^2} + \frac{\mathrm{P}\text{\textsc{n}} - \text{\textsc{p}}\mathrm{N}}{\mathrm{N}^2 + \text{\textsc{n}}^2}.\, \frac{\cos.\varphi}{\sin.\varphi}$$

$$\text{\textsc{a}} = - \frac{\mathrm{P}\text{\textsc{n}} + \text{\textsc{p}}\mathrm{N}}{\mathrm{N}^2 + \text{\textsc{n}}^2}.\, \frac{q}{p.\, \sin.\varphi}.$$

208. Inventis ergo hoc modo A & $\text{\textsc{a}}$, fiet $\frac{M - (\mathrm{A} + \text{\textsc{a}}z)Z}{pp - 2pqz.\, \cos.\varphi + qqzz}$ Functio integra, quæ ſit $= P$; atque ſupereſt ut $P - \mathrm{B}Z - \text{\textsc{b}}Zz$ diviſibile evadat per $pp - 2pqz.\, \cos.\varphi + qqzz$, quæ expreſſio cum ſimilis ſit præcedenti, ſi

poſito $z^n = \frac{p^n}{q^n}.\, \cos.\, n\varphi$, vocetur $P = \mathrm{R}$,

&, poſito $z^n = \frac{p^n}{q^n}.\, \sin.\, n\varphi$, vocetur $P = \text{\textsc{r}}$; erit

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$$\mathrm{B} = \frac{\mathrm{RN} + \mathrm{rn}}{\mathrm{N}^2 + \mathrm{n}^2} + \frac{\mathrm{Rn} - \mathrm{rN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{cof.\,\varphi}{fin.\,\varphi}$$

$$\mathrm{b} = - \frac{\mathrm{Rn} + \mathrm{rN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{q}{p.\,fin.\,\varphi}.$$

209. Hinc jam generaliter concludere licet quomodo reſolutio inſtitui debeat, ſi denominator Functionis propoſitæ $\frac{M}{N}$, Factorem habeat $(pp - 2pqz.\,cof.\,\varphi + qqzz)^k$: ſit enim $N = (pp - 2pqz.\,cof.\,\varphi + qqzz)^k Z$, ita ut hæc reſolvenda ſit Functio fracta

$$\frac{M}{(pp - 2pqz.\,cof.\,\varphi + qqzz)^k Z}.$$

Præbeat ergo Factor denominatoris $(pp - 2pqz.\,cof.\,\varphi + qqzz)^k$ has partes:

$$\frac{\mathrm{A} + \mathrm{a}z}{(pp - 2pqz.\,cof.\,\varphi + qqzz)^k} + \frac{\mathrm{B} + \mathrm{b}z}{(pp - 2pqz.\,cof.\,\varphi + qqzz)^{k-1}} + \frac{\mathrm{C} + \mathrm{c}z}{(pp - 2pqz.\,cof.\,\varphi + qqzz)^{k-2}} + \frac{\mathrm{D} + \mathrm{d}z}{(pp - 2pqz.\,cof.\,\varphi + qqzz)^{k-3}} + \&c.$$

Jam, poſito $z^n = \frac{p^n}{q^n}.\,cof.\,n\varphi$, ſit $M = \mathrm{M}$, & $Z = \mathrm{N}$;

&, poſito $z^n = \frac{p^n}{q^n}.\,fin.\,n\varphi$, ſit $M = \mathrm{m}$, & $Z = \mathrm{n}$;

erit

$$\mathrm{A} = \frac{\mathrm{MN} + \mathrm{mn}}{\mathrm{N}^2 + \mathrm{n}^2} + \frac{\mathrm{Mn} - \mathrm{mN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{cof.\,\varphi}{fin.\,\varphi}$$

$$\mathrm{a} = - \frac{\mathrm{Mn} + \mathrm{mN}}{\mathrm{N}^2 + \mathrm{n}^2} \cdot \frac{q}{p.\,fin.\,\varphi}.$$

Deinde vocetur $\frac{M - (\mathrm{A} + \mathrm{a}z) Z}{pp - 2pqz.\,cof.\,\varphi + qqzz} = P$; atque;

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poſito $z^n = \frac{p^n}{q^n} . \, coſ. \, n\varphi$, ſit $P = \mathrm{P}$,

&, poſito $z^n = \frac{p^n}{q^n} . \, ſin. \, n\varphi$, ſit $P = \mathfrak{p}$;

erit

$$\mathrm{B} = \frac{\mathrm{PN} + \mathfrak{p}\mathfrak{n}}{\mathrm{N}^2 + \mathfrak{n}^2} + \frac{\mathrm{P}\mathfrak{n} - \mathfrak{p}\mathrm{N}}{\mathrm{N}^2 + \mathfrak{n}^2} \cdot \frac{coſ. \, \varphi}{ſin. \, \varphi}$$

$$\mathfrak{b} = - \frac{\mathrm{P}\mathfrak{n} + \mathfrak{p}\mathrm{N}}{\mathrm{N}^2 + \mathfrak{n}^2} \cdot \frac{q}{p . \, ſin. \varphi} .$$

Tum vocetur $\frac{P - (\mathrm{B} + \mathfrak{b}z) Z}{pp - 2pqz . \, coſ. \, \varphi + qqzz} = Q$, atque

poſito $z^n = \frac{p^n}{q^n} . \, coſ. \, n\varphi$, ſit $Q = \mathrm{Q}$,

&, poſito $z^n = \frac{p^n}{q^n} . \, ſin. \, n\varphi$, ſit $Q = \mathfrak{q}$;

erit

$$\mathrm{C} = \frac{\mathrm{QN} + \mathfrak{q}\mathfrak{n}}{\mathrm{N}^2 + \mathfrak{n}^2} + \frac{\mathrm{Q}\mathfrak{n} - \mathfrak{q}\mathrm{N}}{\mathrm{N}^2 + \mathfrak{n}^2} \cdot \frac{coſ. \, \varphi}{ſin. \, \varphi}$$

$$\mathfrak{c} = - \frac{\mathrm{Q}\mathfrak{n} + \mathfrak{q}\mathrm{N}}{\mathrm{N}^2 + \mathfrak{n}^2} \cdot \frac{q}{p . \, ſin. \varphi} .$$

Porro vocetur $\frac{Q - (\mathrm{C} + \mathfrak{c}z) Z}{pp - 2pqz . \, coſ. \, \varphi + qqzz} = R$, atque

poſito $z^n = \frac{p^n}{q^n} . \, coſ. \, n\varphi$, ſit $R = \mathrm{R}$,

&, poſito $z^n = \frac{p^n}{q^n} . \, ſin. \, n\varphi$, ſit $R = \mathfrak{r}$;

erit

$$\mathrm{D} = \frac{\mathrm{RN} + \mathfrak{r}\mathfrak{n}}{\mathrm{N}^2 + \mathfrak{n}^2} + \frac{\mathrm{R}\mathfrak{n} - \mathfrak{r}\mathrm{N}}{\mathrm{N}^2 + \mathfrak{n}^2} \cdot \frac{coſ. \, \varphi}{ſin. \, \varphi}$$

$$\mathfrak{d} = - \frac{\mathrm{R}\mathfrak{n} + \mathfrak{r}\mathrm{N}}{\mathrm{N}^2 + \mathfrak{n}^2} \cdot \frac{q}{p . \, ſin. \varphi} .$$

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Hocque modo progrediendum eſt donec ultimæ fractionis, cujus denominator eſt $pp - 2pqz. coſ.\varphi + qqzz$, numerator fuerit determinatus. CAP. XII.

EXEMPLUM.

Sit iſta propoſita Functio fracta

$$\frac{z - z^3}{(1+zz)^4(1+z^4)}$$

ex cujus denominatoris Factore $(1+zz)^4$ oriantur hæ fractiones partiales,

$$\frac{\mathrm{A} + \mathfrak{a}z}{(1+zz)^4} + \frac{\mathrm{B} + \mathfrak{b}z}{(1+zz)^3} + \frac{\mathrm{C} + \mathfrak{c}z}{(1+zz)^2} + \frac{\mathrm{D} + \mathfrak{d}z}{1+zz}.$$

Comparatione ergo inſtituta, erit $p=1$, $q=1$, $coſ.\varphi = 0$; ideoque $\varphi = \frac{1}{2}\pi$, porroque $M = z - z^3$ & $Z = 1 + z^4$. Hinc erit $\mathrm{M} = 0$; $\mathfrak{m} = 2$; $\mathrm{N} = 2$; $\mathfrak{n} = 0$, & $ſin.\varphi = 1$.

Hinc itaque invenitur

$$\mathrm{A} = -\frac{4}{4}.0 = 0, \ \& \ \mathfrak{a} = 1.$$

Ergo $\mathrm{A} + \mathfrak{a}z = z$; hincque $P = \frac{z - z^3 - z - z^5}{1+zz} = -z^3$, & $\mathrm{P} = 0$, $\mathfrak{p} = 1$: unde reperitur

$$\mathrm{B} = 0, \ \& \ \mathfrak{b} = \frac{1}{2}.$$

Ergo $\mathrm{B} + \mathfrak{b}z = \frac{1}{2}z$, & $Q = \frac{-z^3 - \frac{1}{2}z - \frac{1}{2}z^5}{1+zz} = -\frac{1}{2}z - \frac{1}{2}z^3$;

unde $\mathrm{Q} = 0$ & $\mathfrak{q} = 0$, ergo

$\mathrm{C} = 0$ & $\mathfrak{c} = 0$, hincque $R = -\frac{\frac{1}{2}z - \frac{1}{2}z^3}{1+zz} = -\frac{1}{2}z$,

ergo $\mathrm{R} = 0$; $\mathfrak{r} = -\frac{1}{2}$; unde fit

$$\mathrm{D} = 0 \ \& \ \mathfrak{d} = -\frac{1}{4}.$$

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 Quamobrem fractiones quæſitæ ſunt hæ $\frac{z}{(1+zz)^3} + \frac{z}{2(1+zz)^2} - \frac{z}{4(1+zz)}$. Reliquæ vero fractionis numerator eſt $= S = \frac{R-(D+\mathfrak{d}z)Z}{1+zz} = -\frac{1}{4}z + \frac{1}{4}z^3$, quæ ergo erit $= \frac{-z+z^3}{4(1+z^2)}$.

210. Hac ergo methodo ſimul innoteſcit fractio complementi, quæ cum inventis conjuncta producat fractionem propoſitam ipſam. Scilicet ſi fractionis

$$\frac{M}{(pp - 2pqz.\,coſ.\,\varphi + qqzz)^k\, Z}$$

inventæ fuerint omnes fractiones partiales ex Factore $(pp - 2pqz.\,coſ.\,\varphi + qqzz)^k$ oriundæ, pro quibus formati ſunt valores Functionum P, Q, R, S, T, ſi harum litterarum Series ulterius continuetur, erit ea, quæ ultimam, qua opus eſt ad numeratores inveniendos, ſequitur, numerator reliquæ fractionis denominatorem Z habentis; nempe, ſi $k = 1$, erit reliqua fractio $\frac{P}{Z}$; ſi $k = 2$, erit reliqua fractio $\frac{Q}{Z}$; ſi $k = 3$, erit ea $\frac{R}{Z}$, & ita porro. Inventa autem hac reliqua fractione denominatorem Z habente, ea per has regulas ulterius reſolvi poterit.

CAPUT XIII.

De Seriebus recurrentibus.

211. Ad hoc Serierum genus, quas Moivræus *recurrentes* vocare ſolet, hic refero omnes Series quæ ex evolutione Functionis cujuſque fractæ per diviſionem actualem inſtituta naſcuntur. Supra enim jam oſtendimus has Series ita eſſe comparatas, ut quivis terminus ex aliquot præcedentibus ſecundum legem quandam conſtantem determinetur, quæ lex a denominatore Functionis fractæ pendet. Cum autem nunc Functionem quamcunque fractam in alias ſimpliciores reſolvere docuerim, hinc Series quoque recurrens in alias ſimpliciores reſolvetur. In hoc igitur Capite propoſitum eſt Serierum recurrentium cujuſvis gradus reſolutionem in ſimpliciores exponere.

212. Sit propoſita iſta Functio fracta genuina

$$\frac{a + bz + czz + dz^3 + \&c.}{1 - \alpha z - \beta zz - \gamma z^3 - \delta z^4 - \&c.}$$

quæ per diviſionem evolvatur in hanc Seriem recurrentem

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus coëfficientes quemadmodum progrediantur, ſupra eſt oſtenſum. Quod ſi jam Functio illa fracta reſolvatur in fractiones ſuas ſimplices, & unaquæque in Seriem recurrentem evolvatur, manifeſtum eſt ſummam omnium harum Serierum ex fractionibus partialibus ortarum æqualem eſſe debere Seriei recurrenti

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.$$

Fractiones ergo partiales, quas ſupra invenire docuimus, dabunt

LIB. I. Series partiales, quarum indoles ob ſimplicitatem facile perſpicitur; omnes autem Series partiales junctim ſumtæ producent Seriem recurrentem propoſitam; unde & hujus natura penitius cognoſcetur.

213. Sint Series recurrentes ex ſingulis fractionibus partialibus ortæ hæ.

$$\begin{array}{l} a + bz + czz + dz^3 + ez^4 + \&c. \\ a' + b'z + c'zz + d'z^3 + e'z^4 + \&c. \\ a'' + b''z + c''zz + d''z^3 + e''z^4 + \&c. \\ a''' + b'''z + c'''zz + d'''z^3 + e'''z^4 + \&c. \\ \&c. \end{array}$$

Quoniam hæ Series junctim ſumtæ æquales eſſe debent huic

$$A + Bz + Czz + Dz^3 + Ez^4 + \&c.$$

neceſſe eſt ut ſit

$$\begin{array}{l} A = a + a' + a'' + a''' + \&c. \\ B = b + b' + b'' + b''' + \&c. \\ C = c + c' + c'' + c''' + \&c. \\ D = d + d' + d'' + d''' + \&c. \\ \&c. \end{array}$$

Hinc, ſi ſingularum Serierum ex fractionibus partialibus ortarum definiri queant coëfficientes Poteſtatis z^n, horum ſumma dabit coëfficientem Poteſtatis z^n in Serie recurrente $A + Bz + Cz^2 + Dz^3 + \&c.$

214. Dubium hic ſuboriri poſſet, an, ſi duæ hujuſmodi Series fuerint inter ſe æquales

$A + Bz + Cz^2 + Dz^3 + \&c. = \mathrm{A} + \mathrm{B}z + \mathrm{C}z^2 + \mathrm{D}z^3 + \&c.$, neceſſario inde ſequatur, coëfficientes ſimilium Poteſtatum ipſius z inter ſe eſſe æquales; ſeu an ſit $A = \mathrm{A}$; $B = \mathrm{B}$; $C = \mathrm{C}$;

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$C = \mathrm{C}$; $D = \mathrm{D}$; &c.. Hoc autem dubium facile tolletur, ſi perpendamus hanc æqualitatem ſubſiſtere debere quemcunque valorem obtineat variabilis z. Sit igitur $z = 0$, atque manifeſtum eſt fore $A = \mathrm{A}$. His ergo terminis æqualibus utrinque ſublatis, ac reliqua æquatione per z diviſa, habebitur $B + Cz + Dz^2 + \&c. = \mathrm{B} + \mathrm{C}z + \mathrm{D}z^2 + \&c.$; unde ſequitur fore $B = \mathrm{B}$: ſimili autem modo oſtendetur eſſe $C = \mathrm{C}$; $D = \mathrm{D}$, & ita porro in infinitum. CAP. XIII.

215. Contemplemur ergo Series, quæ ex fractionibus partialibus, in quas fractio quæpiam propoſita reſolvitur, oriuntur. Ac primo quidem patet fractionem $\frac{\mathrm{A}}{1-pz}$ dare Seriem $\mathrm{A} + \mathrm{A}pz + \mathrm{A}p^2z^2 + \mathrm{A}p^3z^3 + \&c.$, cujus terminus generalis eſt $\mathrm{A}p^nz^n$; hæc enim expreſſio vocari ſolet *terminus generalis*, quoniam ex ea, loco n numeros omnes ſucceſſive ſubſtituendo, omnes Seriei termini naſcuntur. Deinde ex fractione $\frac{\mathrm{A}}{(1-pz)^2}$ oritur Series $\mathrm{A} + 2\mathrm{A}pz + 3\mathrm{A}p^2z^2 + 4\mathrm{A}p^3z^3 + \&c.$, cujus terminus generalis eſt $(n+1)\mathrm{A}p^nz^n$. Tum ex fractione $\frac{\mathrm{A}}{(1-pz)^3}$ oritur Series $\mathrm{A} + 3\mathrm{A}pz + 6\mathrm{A}p^2z^2 + 10\mathrm{A}p^3z^3 + \&c.$, cujus terminus generalis eſt $\frac{(n+1)(n+2)}{1.\ 2}\mathrm{A}p^nz^n$. Generatim autem fractio $\frac{\mathrm{A}}{(1-pz)^k}$ præbet Seriem hanc $\mathrm{A} + k\mathrm{A}pz + \frac{k(k+1)}{1.\ 2}\mathrm{A}p^2z^2 + \frac{k(k+1)(k+2)}{1.\ 2.\ 3}\mathrm{A}p^3z^3 + \&c.$, cujus terminus generalis eſt $\frac{(n+1)(n+2)(n+3)\ \ldots\ldots\ (n+k-1)}{1.\ \ 2.\ \ 3\ \ldots\ldots\ (k-1)}\mathrm{A}p^nz^n$. Ex ipſa autem Seriei progreſſione colligitur hic idem terminus $= \frac{k(k+1)(k+2)\ \ldots\ldots\ (k+n-1)}{1.\ \ 2.\ \ 3\ \ldots\ldots\ n}\mathrm{A}p^nz^n$: hæc vero

LIB. I. expreſſio illi eſt æqualis, id quod eliminatis denominatoribus patebit, fiet enim,

$$1.2.3 \ldots\ldots n(n+1) \ldots\ldots (n+k-1) = 1.2.3 \ldots\ldots (k-1)k \ldots\ldots (k+n-1)$$

quæ eſt æquatio identica.

216. Quoties ergo in reſolutione Functionum fractarum ad hujuſmodi fractiones partiales $\frac{A}{(1-pz)^k}$ pervenitur, toties Seriei recurrentis ex illa Functione fracta ortæ $A + Bz + Cz^2 + Dz^3 + \&c.$, terminus generalis aſſignari poterit, quippe qui erit ſumma terminorum generalium Serierum, quæ ex fractionibus partialibus naſcuntur.

EXEMPLUM I.

Invenire terminum generalem Seriei recurrentis, quæ ex hac fractione $\frac{1-z}{1-z-2zz}$ *naſcitur.*

Series hinc nata eſt $1 + 0z + 2zz + 2z^3 + 6z^4 + 10z^5 + 22z^6 + 42z^7 + 86z^8 + \&c.$. Ad coëfficientem poteſtatis generalis z^n inveniendum, fractio $\frac{1-z}{1-z-2zz}$ reſolvatur in $\frac{\frac{2}{3}}{1+z} + \frac{\frac{1}{3}}{1-2z}$, unde oritur terminus generalis quæſitus $\left(\frac{2}{3}(-1)^n + \frac{1}{3}\cdot 2^n\right) z^n = \frac{2^n \pm 2}{3} z^n$, ubi ſignum + valet ſi n ſit numerus par, ſignum − ſi n ſit impar.

EXEMPLUM II.

Invenire terminum generalem Seriei recurrentis quæ oritur ex fractione $\frac{1-z}{1-5z+6zz}$, *ſeu Seriei hujus* $1 + 4z + 14z^2 + 46z^3 + 146z^4 + 454z^5 + \&c.$

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Ob denominatorem $= (1 - 2z)(1 - 3z)$ reſolvitur fractio in has $\frac{-1}{1-2z} + \frac{2}{1-3z}$, ex quibus ſit terminus generalis $2.\,3^n z^n - 2^n z^n = (2.\,3^n - 2^n)\, z^n$. CAP. XIII.

EXEMPLUM III.

Invenire terminum generalem Seriei hujus $1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + 29z^6 + 47z^7 +$ &c., *quæ oritur ex evolutione fractionis* $\frac{1+2z}{1-z-zz}$.

Ob denominatoris Factores $1 - \left(\frac{1+\sqrt{5}}{2}\right)z$ & $1 - \left(\frac{1-\sqrt{5}}{2}\right)z$, per reſolutionem prodeunt $\frac{\frac{1+\sqrt{5}}{2}}{1-\left(\frac{1+\sqrt{5}}{2}\right)z} + \frac{\frac{1-\sqrt{5}}{2}}{1-\left(\frac{1-\sqrt{5}}{2}\right)z}$, unde erit terminus generalis $=$ $\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} z^n + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} z^n$.

EXEMPLUM IV.

Invenire terminum generalem Seriei hujus $a + (\alpha a + b)z + (\alpha^2 a + \alpha b + \beta a)z^2 + (\alpha^3 a + \alpha^2 b + 2\alpha\beta a + \beta b)z^3 +$ &c., *quæ oritur ex evolutione fractionis* $\frac{a+bz}{1-\alpha z-\beta zz}$.

Per reſolutionem oriuntur hæ duæ fractiones:

$$\frac{(a(\alpha+\surd(\alpha\alpha+4\beta))+2b):2\surd(\alpha\alpha+4\beta)}{1-(\frac{\alpha+\surd(\alpha\alpha+4\beta)}{2})z}+$$

$$\frac{(a(\surd(\alpha\alpha+4\beta)-\alpha)-2b):2\surd(\alpha\alpha+4\beta)}{1-(\frac{\alpha-\surd(\alpha\alpha+4\beta)}{2})z},$$ hinc

terminus generalis erit $\frac{a(\surd(\alpha\alpha+4\beta)+\alpha)+2b}{2\surd(\alpha\alpha+4\beta)}\times$ $(\frac{\alpha+\surd(\alpha\alpha+4\beta)}{2})^n z^n + \frac{a(\surd(\alpha\alpha+4\beta)-\alpha)-2b}{2\surd(\alpha\alpha+4\beta)}\times$ $(\frac{\alpha-\surd(\alpha\alpha+4\beta)}{2})^n z^n$; ex quo omnium Serierum recurrentium, quarum quiſque terminus per duos præcedentes determinatur, termini generales expedite definiri poterunt.

EXEMPLUM V.

Invenire terminum generalem hujus Seriei $1+z+2z^2+2z^3+3z^4+3z^5+4z^6+4z^7+$ &c., *quæ oritur ex fractione* $\frac{1}{1-z-zz+z^3}=\frac{1}{(1-z)^2(1+z)}$.

Quanquam lex progreſſionis primo intuitu ita eſt manifeſta ut explicatione non indigeat, tamen fractiones per reſolutionem ortæ $\frac{\frac{1}{2}}{(1-z)^2}+\frac{\frac{1}{4}}{1-z}+\frac{\frac{1}{4}}{1+z}$ dant hunc terminum generalem $\frac{1}{2}(n+1)z^n+\frac{1}{4}z^n+\frac{1}{4}(-1)^n z^n = \frac{2n+3\pm1}{4}z^n$, ubi ſignum ſuperius valet ſi n fuerit numerus par, inferius ſi n fuerit impar.

217. Hoc pacto omnium Serierum recurrentium termini generales exhiberi poſſunt, quoniam omnes fractiones in hujuſmodi fractiones partiales ſimplices reſolvere licet. Quod ſi autem expreſſiones imaginarias vitare velimus, ſæpenumero ad hujuſmodi fractiones partiales pervenietur

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$\frac{A + Bpz}{1 - 2pz.\,coſ.\varphi + ppzz}$; $\frac{A + Bpz}{(1 - 2pz.\,coſ.\varphi + ppzz)^2}$; &c. CAP. XIII.

$\frac{A + Bpz}{(1 - 2pz.\,coſ.\varphi + ppzz)^k}$, ex quarum evolutione cujuſmodi Series naſcantur videndum eſt. Ac primo quidem, ob $coſ.\,n\varphi = 2\,coſ.\varphi.\,coſ.(n-1)\varphi - coſ.(n-2)\varphi$, fractio $\frac{A}{1 - 2pz.\,coſ.\varphi + ppzz}$ evoluta dabit

$$\begin{aligned}A &+ 2Apz.\,coſ.\varphi + 2Appzz.\,coſ.2\varphi + 2Ap^3z^3.\,coſ.3\varphi + 2Ap^4z^4.\,coſ.4\varphi\\ &+ Appzz \qquad\qquad + 2Ap^3z^3.\,coſ.\varphi + 2Ap^4z^4.\,coſ.2\varphi\\ &\qquad\qquad\qquad\qquad\qquad\qquad + Ap^4z^4 \quad \&c.\end{aligned}$$

cujus Seriei terminus generalis non tam facile apparet.

218. Quo igitur ad ſcopum perveniamus, conſideremus has duas Series

$$Ppz.\,ſin.\varphi + Pp^2z^2.\,ſin.2\varphi + Pp^3z^3.\,ſin.3\varphi + Pp^4z^4.\,ſin.4\varphi + \&c.$$
$$Q + Qpz.\,coſ.\varphi + Qp^2z^2.\,coſ.2\varphi + Qp^3z^3.\,coſ.3\varphi + Qp^4z^4.\,coſ.4\varphi + \&c.$$

quæ duæ Series utique naſcuntur ex evolutione fractionis, cujus denominator eſt $1 - 2pz.\,coſ.\varphi + ppzz$. Ac prior quidem oritur ex hac fractione $\frac{Ppz.\,ſin.\varphi}{1 - 2pz.\,coſ.\varphi + ppzz}$, poſterior vero ex hac $\frac{Q - Qpz.\,coſ.\varphi}{1 - 2pz.\,coſ.\varphi + ppzz}$. Addantur hæ duæ fractiones, atque ſumma $\frac{Q + Ppz.\,ſin.\varphi - Qpz.\,coſ.\varphi}{1 - 2pz.\,coſ.\varphi + ppzz}$ dabit Seriem cujus terminus generalis erit $= (P\,ſin.\,n\varphi + Q\,coſ.\,n\varphi)\,p^n z^n$. Fiat autem hæc fractio propoſitæ $\frac{A + Bpz}{1 - 2pz.\,coſ.\varphi + ppzz}$ æqualis, erit $Q = A$, & $P = A\,cot.\,\varphi + B\,coſec.\,\varphi$. Seriei ergo ex hac fractione $\frac{A + Bpz}{1 - 2pz.\,coſ.\varphi + ppzz}$ ortæ terminus generalis erit $= \frac{A\,coſ.\varphi.\,ſin.\,n\varphi + B\,ſin.\,n\varphi + A\,ſin.\varphi.\,coſ.\,n\varphi}{ſin.\varphi}\,p^n z^n = \frac{A\,ſin.(n+1)\varphi + B\,ſin.\,n\varphi}{ſin.\varphi}\,p^n z^n$.

LIB. I. 219. Ad terminum generalem inveniendum, ſi denominator fractionis fuerit Poteſtas, ut $(1 - 2pz.\cos.\varphi + ppzz)^k$, conveniet hanc fractionem reſolvi in duas etſi imaginarias

$$\frac{a}{(1-(\cos.\varphi+\sqrt{-1}.\sin.\varphi)pz)^k} + \frac{b}{(1-(\cos.\varphi-\sqrt{-1}.\sin.\varphi)pz)^k},$$

quarum ſimul ſumtarum terminus generalis Seriei ex ipſis ortæ erit

$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1.\ \ 2.\ \ 3\ \ldots\ldots\ (k-1)}(\cos.n\varphi+\sqrt{-1}.\sin.n\varphi)ap^nz^n +$$
$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1.\ \ 2.\ \ 3\ \ldots\ldots\ (k-1)}(\cos.n\varphi-\sqrt{-1}.\sin.n\varphi)bp^nz^n.$$

Sit $a + b = f$; $a - b = \frac{g}{\sqrt{-1}}$, ut ſit $a = \frac{f\sqrt{-1}+g}{2\sqrt{-1}}$ & $b = \frac{f\sqrt{-1}-g}{2\sqrt{-1}}$, eritque hæc expreſſio

$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1.\ \ 2.\ \ 3\ \ldots\ldots\ (k-1)}(f.\cos.n\varphi + g.\sin.n\varphi)p^nz^n$$

terminus generalis Seriei, quæ oritur ex his fractionibus

$$\frac{\frac{1}{2}f + \frac{1}{2\sqrt{-1}}g}{(1-(\cos.\varphi+\sqrt{-1}.\sin.\varphi)pz)^k} + \frac{\frac{1}{2}f - \frac{1}{2\sqrt{-1}}g}{(1-(\cos.\varphi-\sqrt{-1}.\sin.\varphi)pz)^k},$$

ſeu quæ oritur ex hac fractione una

$$\frac{\begin{array}{l} f - kfpz.\cos.\varphi + \frac{k(k-1)}{1.\ 2}fp^2z^2.\cos.2\varphi - \frac{k(k-1)(k-2)}{1.\ 2.\ 3}fp^3z^3.\cos.3\varphi \\ + kgpz.\sin.\varphi - \frac{k(k-1)}{1.\ 2}gp^2z^2.\sin.2\varphi + \frac{k(k-1)(k-2)}{1.\ 2.\ 3}gp^3z^3.\sin.3\varphi \end{array}\ \&c.}{(1 - 2pz.\cos.\varphi + ppzz)^k}$$

220. Poſito ergo $k = 2$, erit Seriei ex hac fractione

$$\frac{f - 2pz(f.\cos.\varphi - g.\sin.\varphi) + ppzz(f.\cos.2\varphi - g.\sin.2\varphi)}{(1 - 2pz.\cos.\varphi + ppzz)^2}$$

ortæ terminus generalis $= (n+1)(f.\cos.n\varphi + g.\sin.n\varphi)p^nz^n$. At Seriei ex hac fractione $\frac{a}{1 - 2pz.\cos.\varphi + ppzz}$, ſeu hac

$\frac{a - 2apz.\,\text{coſ.}\,\varphi + appzz}{(1 - 2pz.\,\text{coſ.}\,\varphi + ppzz)^2}$ ortæ terminus generalis eſt $= \frac{a.\,\text{ſin.}\,(n+1)\varphi}{\text{ſin.}\,\varphi}\,p^n z^n$. Addantur hæ fractiones invicem, ac ponatur $a + f = \text{A}$; $2a.\,\text{coſ.}\,\varphi + 2f.\,\text{coſ.}\,\varphi - 2g.\,\text{ſin.}\,\varphi = -\text{B}$ & $a + f.\,\text{coſ.}\,2\varphi - g.\,\text{ſin.}\,2\varphi = 0$, hinc erit $g = \frac{\text{B} + 2\text{A}\,\text{coſ.}\,\varphi}{2\,\text{ſin.}\,\varphi}$, $a = \frac{\text{A} + \text{B}\,\text{coſ.}\,\varphi}{1 - \text{coſ.}\,2\varphi} = \frac{\text{A} + \text{B}\,\text{coſ.}\,\varphi}{2(\text{ſin.}\,\varphi)^2}$ & $f = -\frac{\text{A}\,\text{coſ.}\,2\varphi - \text{B}\,\text{coſ.}\,\varphi}{2(\text{ſin.}\,\varphi)^2}$, & $g = \frac{\text{B}\,\text{ſin.}\,\varphi + \text{A}\,\text{ſin.}\,2\varphi}{2(\text{ſin.}\,\varphi)^2}$. Hanc ob rem Seriei ex hac fractione $\frac{\text{A} + \text{B}pz}{(1 - 2pz.\,\text{coſ.}\,\varphi + ppzz)^2}$, ortæ terminus generalis eſt $\frac{\text{A} + \text{B}\,\text{coſ.}\,\varphi}{2(\text{ſin.}\,\varphi)^2}\,\text{ſin.}\,(n+1)\varphi.\,p^n z^n + (n+1) \times$
$\frac{(\text{B}\,\text{ſin.}\,\varphi.\,\text{ſin.}\,n\varphi + \text{A}\,\text{ſin.}\,2\varphi.\,\text{ſin.}\,n\varphi - \text{B}\,\text{coſ.}\,\varphi.\,\text{coſ.}\,n\varphi - \text{A}\,\text{coſ.}\,2\varphi.\,\text{coſ.}\,n\varphi)}{2(\text{ſin.}\,\varphi)^2} \times$
$p^n z^n = -\frac{(n+1)(\text{A}\,\text{coſ.}\,(n+2)\varphi + \text{B}\,\text{coſ.}\,(n+1)\varphi)}{2(\text{ſin.}\,\varphi)^2}\,p^n z^n +$
$\frac{(\text{A} + \text{B}\,\text{coſ.}\,\varphi)\,\text{ſin.}\,(n+1)\varphi}{2(\text{ſin.}\,\varphi)^2}\,p^n z^n =$
$\frac{(\frac{1}{2}(n+3)\,\text{ſin.}\,(n+1)\varphi - \frac{1}{2}(n+1)\,\text{ſin.}\,(n+3)\varphi)}{2(\text{ſin.}\,\varphi)^3}\,\text{A}\,p^n z^n +$
$\frac{(\frac{1}{2}(n+2)\,\text{ſin.}\,n\varphi - \frac{1}{2}n\,\text{ſin.}\,(n+2)\varphi)}{2(\text{ſin.}\,\varphi)^3}\,\text{B}\,p^n z^n$. Eſt ergo iſte terminus generalis quæſitus $=$
$\frac{(n+3)\,\text{ſin.}\,(n+1)\varphi - (n+1)\,\text{ſin.}\,(n+3)\varphi}{4(\text{ſin.}\,\varphi)^3}\,\text{A}\,p^n z^n +$
$\frac{(n+2)\,\text{ſin.}\,n\varphi - n.\,\text{ſin.}\,(n+2)\varphi}{4(\text{ſin.}\,\varphi)^3}\,\text{B}\,p^n z^n$ Seriei quæ oritur ex fractione $\frac{\text{A} + \text{B}pz}{(1 - 2pz.\,\text{coſ.}\,\varphi + ppzz)^2}$.

221. Sit $k = 3$, eritque Seriei ex hac fractione ortæ
$\frac{f - 3pz(f.\,\text{coſ.}\,\varphi - g.\,\text{ſin.}\,\varphi) + 3ppzz(f.\,\text{coſ.}\,2\varphi - g.\,\text{ſin.}\,2\varphi) - p^3z^3(f.\,\text{coſ.}\,3\varphi - g.\,\text{ſin.}\,3\varphi)}{(1 - 2pz.\,\text{coſ.}\,\varphi + ppzz)^3}$
terminus generalis $= \frac{(n+1)(n+2)}{1.\quad 2}\,(f.\,\text{coſ.}\,n\varphi + g.\,\text{ſin.}\,n\varphi)\,p^n z^n$.

Deinde Seriei ex fractione $\frac{a+bpz}{(1-2pz\cos\varphi+ppzz)^3}$, ſeu ex hac $\frac{a-2apz\cos\varphi+appzz+bpz-2bppzz\cos\varphi+bp^3z^3}{(1-2pz\cos\varphi+ppzz)^4}$ ortæ terminus generalis eſt $\frac{(n+3)\sin(n+1)\varphi-(n+1)\sin(n+3)\varphi}{4(\sin\varphi)^3}ap^nz^n+\frac{(n+2)\sin n\varphi-n\sin(n+2)\varphi}{4(\sin\varphi)^3}bp^nz^n$. Addantur hæ fractiones ac ponatur numerator $=A$, erit $a+f=A$, $3f\cos\varphi-3g\sin\varphi+2a\cos\varphi-b=0$, $3f\cos 2\varphi-3g\sin 2\varphi+a-2b\cos\varphi=0$ & $b=f\cos 3\varphi-g\sin 3\varphi$, hinc erit $a=\frac{f\cos 3\varphi-g\sin 3\varphi-3f\cos\varphi+3g\sin\varphi}{2\cos\varphi}=2g(\sin\varphi)^2\tan\varphi-f-2f(\sin\varphi)^2$. Deinde reperitur $\frac{f}{g}=\frac{\sin 5\varphi-2\sin 3\varphi+\sin\varphi}{\cos 5\varphi-2\cos 3\varphi+\cos\varphi}$ & $a+f=A=2g(\sin\varphi)^2\tan\varphi-2f(\sin\varphi)^2$; ergo $\frac{A}{2(\sin\varphi)^2}=\frac{g\sin\varphi-f\cos\varphi}{\cos\varphi}$; ex quibus tandem oritur $f=\frac{A(\sin\varphi-2\sin 3\varphi+\sin 5\varphi)}{16(\sin\varphi)^5}$, $g=\frac{A(\cos\varphi-2\cos 3\varphi+\cos 5\varphi)}{16(\sin\varphi)^5}$, ob $16(\sin\varphi)^5=\sin 5\varphi-5\sin 3\varphi+10\sin\varphi$, erit $a=\frac{A(9\sin\varphi-3\sin 3\varphi)}{16(\sin\varphi)^5}$ & $b=\frac{A(-\sin 2\varphi+\sin 2\varphi)}{16(\sin\varphi)^5}=0$. Eſt autem $3\sin\varphi-\sin 3\varphi=4(\sin\varphi)^3$; ergo $a=\frac{3A}{4(\sin\varphi)^2}$. Quocirca erit terminus generalis $\frac{(n+1)(n+2)}{1\cdot 2}p^nz^n\times$

$$A\frac{(\sin(n+1)\varphi-2\sin(n+3)\varphi+\sin(n+5)\varphi)}{16(\sin\varphi)^5}+$$

$$3Ap^nz^n\cdot\frac{((n+3)\sin(n+1)\varphi-(n+1)\sin(n+3)\varphi)}{16(\sin\varphi)^5}=$$

$$\frac{Ap^nz^n}{16(\sin\varphi)^5}\left(\frac{(n+4)(n+5)}{1\cdot 2}\sin(n+1)\varphi-\frac{2(n+1)(n+5)}{1\cdot 2}\times\sin(n+3)\varphi+\frac{(n+1)(n+2)}{1\cdot 2}\sin(n+5)\varphi\right).$$

222. Seriei

222. Seriei ergo quæ oritur ex hac fractione

$$\frac{A + Bpz}{(1 - 2pz.\ \mathrm{coſ}.\ \varphi + ppzz)^3}$$

terminus generalis erit hic

$$\frac{Ap^n z^n}{16\,(\mathrm{ſin}.\ \varphi)^5}\left(\frac{(n+5)(n+4)}{1.\ \ 2}\,\mathrm{ſin}.(n+1)\varphi - \frac{2(n+1)(n+5)}{1.\ \ 2}\times\right.$$
$$\left.\mathrm{ſin}.(n+3)\varphi + \frac{(n+1)(n+2)}{1.\ \ 2}\,\mathrm{ſin}.(n+5)\varphi\right)$$
$$+\frac{Bp^n z^n}{16\,(\mathrm{ſin}.\ \varphi)^5},\left(\frac{(n+4)(n+3)}{1.\ \ 2}\,\mathrm{ſin}.\,n\varphi - \frac{2n(n+4)}{1.\ \ 2}\,\mathrm{ſin}.(n+2)\varphi +\right.$$
$$\left.\frac{n(n+1)}{1.\ \ 2}\,\mathrm{ſin}.(n+4)\varphi\right).$$

Atque, ulterius progrediendo, Seriei, quæ oritur ex hac fractione

$$\frac{A + Bpz}{(1 - 2pz.\ \mathrm{coſ}.\ \varphi + ppzz)^4}$$

terminus generalis erit hic

$$+\frac{Ap^n z^n}{64\,(\mathrm{ſin}.\ \varphi)^7},\left(\frac{(n+7)(n+6)(n+5)}{1.\ \ 2.\ \ 3}\,\mathrm{ſin}.(n+1)\varphi -\right.$$
$$\frac{3(n+1)(n+7)(n+6)}{1.\ \ 2.\ \ 3}\,\mathrm{ſin}.(n+3)\varphi + \frac{3(n+1)(n+2)(n+7)}{1.\ \ 2.\ \ 3}\times$$
$$\left.\mathrm{ſin}.(n+5)\varphi - \frac{(n+1)(n+2)(n+3)}{1.\ \ 2.\ \ 3}\,\mathrm{ſin}.(n+7)\varphi\right)$$
$$+\frac{Bp^n z^n}{64\,(\mathrm{ſin}.\ \varphi)^7}\left(\frac{(n+6)(n+5)(n+4)}{1.\ \ 2.\ \ 3}\,\mathrm{ſin}.\,n\varphi -\right.$$
$$\frac{3n(n+6)(n+5)}{1.\ \ 2.\ \ 3}\,\mathrm{ſin}.(n+2)\varphi + \frac{3n(n+1)(n+6)}{1.\ \ 2.\ \ 3}\times$$
$$\left.\mathrm{ſin}.(n+4)\varphi - \frac{n(n+1)(n+2)}{1.\ \ 2.\ \ 3}\,\mathrm{ſin}.(n+6)\varphi\right).$$

Ex his autem expreſſionibus facile intelligitur, quemadmodum formæ terminorum generalium pro altioribus dignitatibus progrediantur. Ad naturam vero harum expreſſionum penitius inſpiciendam notari convenit eſſe

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LIB. I.

$$\text{ſin.}\varphi = \text{ſin.}\varphi$$
$$4(\text{ſin.}\varphi)^3 = 3\text{ſin.}\varphi - \text{ſin.}3\varphi$$
$$16(\text{ſin.}\varphi)^5 = 10\text{ſin.}\varphi - 5\text{ſin.}3\varphi + \text{ſin.}5\varphi$$
$$64(\text{ſin.}\varphi)^7 = 35\text{ſin.}\varphi - 21\text{ſin.}3\varphi + 7\text{ſin.}5\varphi - \text{ſin.}7\varphi$$
$$256(\text{ſin.}\varphi)^9 = 126\text{ſin.}\varphi - 84\text{ſin.}3\varphi + 36\text{ſin.}5\varphi - 9\text{ſin.}7\varphi + \text{ſin.}9\varphi$$
&c.

223. Cum igitur hoc pacto omnes functiones fractæ in fractiones partiales reales reſolvi queant, ſimul omnium Serierum recurrentium termini generales per expreſſiones reales exhiberi poterunt. Quod quo clarius appareat, exempla ſequentia adjuncta ſunt.

EXEMPLUM I.

Ex fractione $\frac{1}{(1-z)(1-zz)(1-z^3)} = \frac{1}{1-z-zz+z^4+z^5-z^6}$, oritur iſta Series recurrens $1+z+2z^2+3z^3+4z^4+5z^5+7z^6+8z^7+10z^8+12z^9+\&c.$; cujus terminus generalis deſideratur. Fractio propoſita ſecundum Factores ordinata fit $= \frac{1}{(1-z)^3(1+z)(1+z+zz)}$, quæ reſolvitur in has fractiones $\frac{1}{6(1-z)^3} + \frac{1}{4(1-z)^2} + \frac{17}{72(1-z)} + \frac{1}{8(1+z)} + \frac{(2+z)}{9(1+z+zz)}$. Harum prima $\frac{1}{6(1-z)^3}$ dat terminum generalem $\frac{(n+1)}{1}\frac{(n+2)}{2}\cdot\frac{1}{6}z^n = \frac{nn+3n+2}{12}z^n$: ſecunda $\frac{1}{4(1-z)^2}$ dat $\frac{n+1}{4}z^n$: tertia $\frac{17}{72(1-z)}$ dat $\frac{17}{72}z^n$: quarta $\frac{1}{8(1+z)}$ dat $\frac{1}{8}(-1)^n z^n$.

Quinta vero $\frac{2+z}{9(1+z+zz)}$ comparata cum forma $\frac{A+Bpz}{1-2pz\cdot\text{coſ.}\,\varphi+ppzz}$ (218) dat $p=1$, $\varphi=\frac{\pi}{3}=60^\circ$; $A=+\frac{2}{9}$ & $B=-\frac{1}{9}$, unde oritur terminus generalis $+\frac{2\,\text{ſin.}(n+1)\varphi-\text{ſin.}\,n\varphi}{9\,\text{ſin.}\,\varphi}(-1)^n z^n=\frac{4\,\text{ſin.}(n+1)\varphi-2\,\text{ſin.}\,n\varphi}{9}\times(-1)^n z^n=\frac{4\,\text{ſin.}(n+1)\frac{\pi}{3}-2\,\text{ſin.}\,n\frac{\pi}{3}}{9}(-1)^n z^n$. Colligantur hæ expreſſiones omnes in unam ſummam, ac prodibit Seriei propoſitæ terminus generalis quæſitus $=(\frac{nn}{12}+\frac{n}{2}+\frac{47}{72})z^n\pm\frac{1}{8}z^n\pm\frac{4\,\text{ſin.}(n+1)\frac{\pi}{3}-2\,\text{ſin.}\,n\frac{\pi}{3}}{9}z^n$, ubi ſigna ſuperiora valent ſi n numerus par, inferiora ſi n impar. Ubi notandum eſt ſi fuerit n numerus formæ $3m$ fore $\frac{4\,\text{ſin.}\frac{1}{3}(n+1)\pi-2\,\text{ſin.}\frac{1}{3}n\pi}{9}=\pm\frac{2}{9}$; ſi fuerit $n=3m+1$ ſeu $n=3m+2$, erit iſta expreſſio $=\mp\frac{1}{9}$, prout n fuerit numerus vel par vel impar. Ex his natura Seriei ita explicari poteſt, ut

LIB. I.

ſi fuerit	terminus generalis futurus ſit
$n = 6m + 0$	$(\frac{nn}{12} + \frac{n}{2} + 1) z^n$
$n = 6m + 1$	$(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12}) z^n$
$n = 6m + 2$	$(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3}) z^n$
$n = 6m + 3$	$(\frac{nn}{12} + \frac{n}{2} + \frac{3}{4}) z^n$
$n = 6m + 4$	$(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3}) z^n$
$n = 6m + 5$	$(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12}) z^n$.

Sic, ſi fuerit $n = 50$, valet forma $n = 6m + 2$, eritque terminus Seriei $= 234 z^{50}$.

EXEMPLUM II.

Ex fractione $\frac{1 + z + zz}{1 - z - z^4 + z^5}$ oritur hæc Series recurrens $1 + 2z + 3zz + 3z^3 + 4z^4 + 5z^5 + 6z^6 + 6z^7 + 7z^8 + \&c.$, cujus terminum generalem invenire oportet. Fractio propoſita ad hanc formam reducitur $\frac{1 + z + zz}{(1 - z)^2 (1 + z)(1 + zz)}$, quæ propterea reſolvitur in has fractiones partiales $\frac{3}{4(1 - z)^2} + \frac{3}{8(1 - z)} + \frac{1}{8(1 + z)} - \frac{1 + z}{4(1 + zz)}$. Harum prima $\frac{3}{4(1 - z)^2}$ dat terminum generalem $\frac{3(n + 1)}{4} z^n$; ſecunda $\frac{3}{8(1 - z)}$ dat $\frac{3}{8} z^n$; tertia dat $\frac{1}{8}(-1)^n z^n$; & quarta $-\frac{1 + z}{4(1 + zz)}$ comparata cum forma $\frac{A + Bpz}{1 - 2pz . cof. \varphi + ppzz}$ dat $p = 1$, $cof. \varphi = 0$ & $\varphi = \frac{1}{2}\pi$; $A = -\frac{1}{4}$,

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$B = +\frac{1}{4}$; unde fit terminus generalis $= (-\frac{1}{4}\,\textit{ſin.}\,\frac{1}{2} \times (n+1)\pi + \frac{1}{4}\,\textit{ſin.}\,\frac{1}{2}n\pi)\,z^n$. Quare colligendo erit terminus generalis quæſitus $= (\frac{3}{4}n + \frac{9}{8})\,z^n \pm \frac{1}{8}\,z^n - \frac{1}{4}\,(\textit{ſin.}\,\frac{1}{2}(n+1)\pi - \textit{ſin.}\,\frac{1}{2}n\pi)\,z^n$. Hinc

ſi fuerit	erit terminus generalis
$n = 4m + 0$	$(\frac{3}{4}n + 1)\,z^n$
$n = 4m + 1$	$(\frac{3}{4}n + \frac{5}{4})\,z^n$
$n = 4m + 2$	$(\frac{3}{4}n + \frac{3}{2})\,z^n$
$n = 4m + 3$	$(\frac{3}{4}n + \frac{3}{4})\,z^n$.

Ita, ſi $n = 50$, valebit $n = 4m + 2$, eritque terminus $= 39\,z^{50}$.

224. Propoſita ergo Serie recurrente, quoniam illa fractio unde oritur, facile cognoſcitur, ejus terminus generalis ſecundum præcepta data reperietur. Ex lege autem Seriei recurrentis, qua quiſque terminus ex præcedentibus definitur, ſtatim innoteſcit denominator fractionis, hujuſque Factores præbebunt formam termini generalis, per numeratorem enim tantum coëfficientes determinantur. Sit nempe propoſita hæc Series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus lex progreſſionis, qua unuſquiſque terminus ex aliquot præcedentibus determinatur, præbeat hunc fractionis denominatorem $1 - \alpha z - \beta z^2 - \gamma z^3$. Ita ut ſit $D = \alpha C + \beta B + \gamma A$; $E = \alpha D + \beta C + \gamma B$; $F = \alpha E + \beta D + \gamma C$;

Lib. I. &c., qui multiplicatores α, $+\beta$, $+\gamma$ a Moivræo *ſcalam relationis* conſtituere dicuntur. Lex ergo progreſſionis poſita eſt in ſcala relationis, atque ſcala relationis ſtatim præbet denominatorem fractionis, ex cujus reſolutione propoſita Series recurrens oritur.

225. Ad terminum ergo generalem, ſeu coëfficientem Poteſtatis indefinitæ z^n, inveniendum, quæri debent denominatoris $1 - \alpha z - \beta z^2 - \gamma z^3$ Factores vel ſimplices vel duplices, ſi imaginarios vitare velimus. Sint primo Factores ſimplices omnes inter ſe inæquales & reales hi $(1-pz)(1-qz)(1-rz)$; atque fractio generans Seriem propoſitam reſolvetur in $\frac{A}{1-pz} + \frac{B}{1-qz} + \frac{C}{1-rz}$; unde Seriei terminus generalis erit $(Ap^n + Bq^n + Cr^n)z^n$. Si duo Factores fuerint æquales nempe $q = p$, tum terminus generalis hujuſmodi erit $((An + B)p^n + Cr^n)z^n$, &, ſi inſuper fuerit $r = q = p$, erit terminus generalis $(An^2 + Bn + C)p^n z^n$. Quod ſi vero denominator $1 - \alpha z - \beta z^2 - \gamma z^3$ duplicem habeat Factorem, ut ſit $= (1-pz)(1 - 2qz.\,coſ.\,\varphi + qqzz)$ tum terminus generalis erit $= (Ap^n + \frac{B\,ſin.(n+1)\varphi + C\,ſin.\,n\varphi}{ſin.\,\varphi} q^n)z^n$. Cum igitur, poſitis pro n ſucceſſive numeris 0, 1, 2, prodire debeant termini A, Bz, Cz^2, hinc valores litterarum A, B, C determinabuntur.

226. Sit ſcala relationis bimembris, ſeu determinetur quiſque terminus per duos præcedentes, ita ut ſit $C = \alpha B - \beta A$; $D = \alpha C - \beta B$; $E = \alpha D - \beta C$, &c., atque manifeſtum eſt Seriem hanc recurrentem, quæ ſit $A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots + Pz^n + Qz^{n+1} + \&c.$,

oriri ex fractione cujus denominator ſit $1 - \alpha z + \beta zz$. Sint hujus denominatoris Factores $(1 - pz)(1 - qz)$, erit $p + q = \alpha$ & $pq = \beta$: atque Seriei terminus generalis erit $(Ap^n + Bq^n) z^n$. Hinc facto $n = 0$, erit $A = \mathrm{A} + \mathrm{B}$; & facto $n = 1$, erit $B = \mathrm{A}p + \mathrm{B}q$; unde ſit $Aq - B = \mathrm{A}(q - p)$ & $\mathrm{A} = \frac{Aq - B}{q - p}$; & $\mathrm{B} = \frac{Ap - B}{p - q}$. Inventis autem valoribus A & B, erit $P = \mathrm{A}p^n + \mathrm{B}q^n$ & $Q = \mathrm{A}p^{n+1} + \mathrm{B}q^{n+1}$. Tum vero erit $\mathrm{AB} = \frac{BB - \alpha AB + \beta AA}{4\beta - \alpha\alpha}$. CAP. XIII.

227. Hinc deduci poteſt modus quemvis terminum ex unico præcedente formandi, cum ad hoc per legem progreſſionis duo requirantur. Cum enim ſit

$$P = \mathrm{A}p^n + \mathrm{B}q^n \ \& \ Q = \mathrm{A}p.p^n + \mathrm{B}q.q^n$$

erit

$$Pq - Q = \mathrm{A}(q - p)p^n \ \& \ Pp - Q = \mathrm{B}(p - q)q^n:$$

multiplicentur hæ expreſſiones in ſe invicem; eritque

$$P^2pq - (p+q)PQ + QQ + \mathrm{AB}(p+q)^2p^nq^n = 0.$$

At eſt

$p + q = \alpha$; $pq = \beta$; $(p - q)^2 = (p + q)^2 - 4pq = \alpha\alpha - 4\beta$ & $p^nq^n = \beta^n$. Quibus ſubſtitutis habebitur $\beta P^2 - \alpha PQ + QQ = (\beta AA + \alpha AB - BB)\beta^n$, ſeu $\frac{QQ - \alpha PQ + \beta PP}{BB - \alpha AB + \beta AA} = \beta^n$; quæ eſt inſignis proprietas Serierum recurrentium, quarum quiſque terminus per duos præcedentes determinatur. At cognito quovis termino P, erit ſequens $Q = \frac{1}{2}\alpha P + \surd((\frac{1}{4}\alpha^2 - \beta)P^2 + (B^2 - \alpha AB + \beta AA)\beta^n)$, quæ expreſſio, etſi ſpeciem irrationalitatis præ ſe

 fert, tamen ſemper eſt rationalis, propterea quod termini irrationales in Serie non occurrunt.

228. Ex datis porro duobus terminis contiguis quibuſvis Pz^n & Qz^{n+1} commode aſſignari poteſt terminus multo magis remotus Xz^{2n}. Ponatur enim

$X = fP^2 + gPQ - hAB\beta^n$. Quoniam eſt

$P = Ap^n + Bq^n$ & $Q = Ap.p^n + Bq.q^n$ atque

$X = Ap^{2n} + Bq^{2n}$; erit ut ſequitur

$$fP^2 = fA^2p^{2n} + fB^2q^{2n} + 2fAB\beta^n$$
$$gPQ = gA^2p.p^{2n} + gB^2q.q^{2n} + gAB\alpha\beta^n$$
$$-hAB\beta^n = \qquad\qquad - hAB\beta^n$$
$$X = Ap^{2n} + Bq^{2n}.$$

Fiet ergo $f + gp = \frac{1}{A}$; $f + gq = \frac{1}{B}$ & $h = 2f + g\alpha$, unde $g = \frac{B - A}{AB(p - q)}$ & $f = \frac{Ap - Bq}{AB(p - q)}$. At eſt $B - A = \frac{\alpha A - 2B}{p - q}$; $Ap - Bq = \frac{\alpha B - 2A\beta}{p - q}$. Ergo $f = \frac{\alpha B - 2A\beta}{AB(\alpha\alpha - 4\beta)}$ & $g = \frac{\alpha A - 2B}{AB(\alpha\alpha - 4\beta)}$ ſeu $f = \frac{2A\beta - \alpha B}{BB - \alpha AB + \beta AA}$ & $g = \frac{2B - \alpha A}{BB - \alpha AB + \beta AA}$; ideoque $h = \frac{(4\beta - \alpha\alpha)A}{BB - \alpha AB + \beta AA}$.

Eritque ergo

$$X = \frac{(2A\beta - \alpha B)P^2 + (2B - \alpha A)PQ}{BB - \alpha AB + \beta AA} - A\beta^n.$$

Simili vero modo reperitur

$$X = \frac{(\alpha\beta A - (\alpha\alpha - 2\beta)B)P^2 + (2B - \alpha A)Q^2}{\alpha(BB - \alpha AB + \beta AA)} - \frac{2B\beta^n}{\alpha}.$$

His conjungendis per eliminationem termini β^n reperitur

$$X = \frac{(\beta A - \alpha B)P^2 + 2BPQ - AQQ}{BB - \alpha AB + \beta AA}.$$

229. Simili

229. Similì modo, ſi ſtatuantur termini ſequentes CAP. XIII.

$A + Bz + Cz^2 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + \ldots\ldots + Xz^{2n} + Yz^{2n+1} + Zz^{2n+2}$,

erit

$$Z = \frac{(\beta A - \alpha B)Q^2 + 2BQR - ARR}{BB - \alpha AB + \beta AA}, \text{ \&, ob } R = \alpha Q - \beta P,$$

erit

$$Z = \frac{-\beta\beta AP^2 + 2\beta(\alpha A - B)PQ + (\alpha B - (\alpha\alpha - \beta)A)Q^2}{BB - \alpha AB + \beta AA}.$$

At eſt

$Z = \alpha Y - \beta X$, ergo $Y = \frac{Z + \beta X}{\alpha}$; unde fit

$Y = \frac{-\beta BP^2 + 2\beta APQ + \alpha(B - \alpha A)QQ}{BB - \alpha AB + \beta AA}$. Sic igitur porro ex X & Y definiri poterunt ſimili modo coëfficientes poteſtatum z^{4n} & z^{4n+1}; hincque ipſarum z^{8n}, z^{8n+1}, & ita porro.

EXEMPLUM.

Sit propoſita iſta Series recurrens

$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \ldots\ldots + Pz^n + Qz^{n+1} + \&c.$,

cujus cum quilibet coëfficiens ſit ſumma duorum præcedentium, erit denominator fractionis hanc Seriem producentis $1 - z - zz$; ideoque $\alpha = 1$; $\beta = -1$; $A = 1$; $B = 3$; unde fit $BB - \alpha AB + \beta AA = 5$; ex quo orietur primum $Q = \frac{P + \sqrt{(5PP + 20(-1)^n)}}{2} = \frac{P + \sqrt{(5PP \pm 20)}}{2}$, ubi ſignum ſuperius valet, ſi n ſit numerus par, inferius ſi impar. Sic, ſi $n = 4$, ob $P = 11$, erit

 $Q = \frac{11 + \surd(5.121 + 20)}{2} = \frac{11 + 25}{2} = 18$. Si porro coëfficiens termini z^{2n} ſit X, erit $X = \frac{-4PP + 6PQ - QQ}{5}$; ergo Poteſtatis z^8 coëfficiens erit $= \frac{-4.121 + 6.198 - 324}{5} = 76$. Cum autem ſit $Q = \frac{P + \surd(5PP \pm 20)}{2}$ erit $QQ = \frac{3PP \pm 10 + P\surd(5PP \pm 20)}{2}$; ideoque $X = \frac{-PP \mp 2 + P\surd(5PP \pm 20)}{2}$. Ex termino ergo Seriei quocunque Pz^n, obtinentur hi $\frac{P + \surd(5PP \pm 20)}{2} z^{n+1}$ & $\frac{-PP \mp 2 + P\surd(5PP \pm 20)}{2} z^{2n}$.

230. Simili modo in Seriebus recurrentibus, quarum quilibet terminus ex tribus antecedentibus determinatur, quivis terminus ex duobus antecedentibus definiri poteſt. Sit enim Series hujuſmodi recurrens
$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + \&c.$, cujus ſcala relationis ſit α, $-\beta$, $+\gamma$, ſeu quæ oriatur ex fractione cujus denominator $= 1 - \alpha z + \beta z^2 - \gamma z^3$. Quod ſi jam termini P, Q, R eodem modo per Factores hujus denominatoris, qui ſint $(1 - pz)(1 - qz)(1 - rz)$ exprimantur, ut ſit $P = \mathrm{A}p^n + \mathrm{B}q^n + \mathrm{C}r^n$, $Q = \mathrm{A}p.p^n + \mathrm{B}q.q^n + \mathrm{C}r.r^n$ & $R = \mathrm{A}p^2.p^n + \mathrm{B}q^2.q^n + \mathrm{C}r^2.r^n$; ob $p + q + r = \alpha$, $pq + pr + qr = \beta$ & $pqr = \gamma$, reperietur hæc proportio

$$R^3 \left.\begin{matrix} -2\alpha Q \\ +\beta P \end{matrix}\right\} R^2 \left.\begin{matrix} +(\alpha\alpha + \beta)\,Q^2 \\ -(\alpha\beta + 3\gamma)\,PQ \\ +\alpha\gamma\,P^2 \end{matrix}\right\} R \begin{matrix} -(\alpha\beta - \gamma)\,Q^3 \\ +(\alpha\gamma + \beta\beta)\,PQ^2 \\ -2\beta\gamma\,P^2Q \\ +\gamma\gamma\,P^3. \end{matrix} : \gamma^n =$$

$$C^3 \left.\begin{matrix} -2\alpha B \\ +\beta A\end{matrix}\right\} C^2 \left.\begin{matrix} +(\alpha^2+\beta)B^2 \\ -(\alpha\beta+3\gamma)AB \\ +\alpha\gamma \quad A^2\end{matrix}\right\} C \begin{matrix} -(\alpha\beta-\gamma)B^3 \\ +(\alpha\gamma+\beta\beta)AB^2 \\ -\quad 2\beta\gamma \quad A^2B \\ +\quad \gamma\gamma \quad A^3\end{matrix} : 1.$$

Pendet ergo inventio termini R ex duobus præcedentibus P & Q a reſolutione æquationis cubicæ.

231. His de terminis generalibus Serierum recurrentium notatis, ſupereſt ut earumdem Serierum ſummas inveſtigemus. Ac primo quidem manifeſtum eſt ſummam Seriei recurrentis in infinitum extenſæ æqualem eſſe fractioni ex qua oritur: cujus fractionis cum denominator ex ipſa progreſſionis lege pateat, reliquum eſt ut numeratorem definiamus. Sit itaque propoſita hæc Series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + Gz^6 + \&c.;$$

cujus lex progreſſionis præbeat hunc denominatorem $1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4$. Sumamus fractionem ſummæ Seriei in infinitum æqualem eſſe $= \frac{a + bz + cz^2 + dz^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$, ex qua cum Series propoſita oriri debeat, erit comparando (63).

$$\begin{aligned} a &= A \\ b &= B - \alpha A \\ c &= C - \alpha B + \beta A \\ d &= D - \alpha C + \beta B - \gamma A: \end{aligned}$$

Hinc erit ſumma quaſita

$$\frac{A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(D - \alpha C + \beta B - \gamma A)z^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

LIB. I. 232. Hinc facile intelligitur quemadmodum Seriei recurrentis ſumma ad datum terminum uſque inveniri debeat. Quæratur ſcilicet Seriei modo aſſumtæ ſumma ad terminum Pz^n, atque ponatur

$$s = A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots\ldots + Pz^n,$$

quoniam hujus Seriei ſumma in infinitum conſtat, quæratur ſumma terminorum ultimum Pz^n in infinitum ſequentium, qui ſint

$$t = Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + Tz^{n+4} + \&c.,$$

hæc Series per z^{n+1} diviſa dat Seriem recurrentem propoſitæ æqualem, cujus propterea ſumma erit $t =$

$$\frac{Qz^{n+1} + (R - \alpha Q)z^{n+2} + (S - \alpha R + \beta Q)z^{n+3} +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(T - \alpha S + \beta R - \gamma Q)z^{n+4}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

Unde orietur ſumma quæſita $s =$

$$\frac{+ A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(D - \alpha C + \beta B - \gamma A)z^3 - Qz^{n+1} -}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(R - \alpha Q)z^{n+2} - (S - \alpha R + \beta Q)z^{n+3} -}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(T - \alpha S + \beta R - \gamma Q)z^{n+4}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

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233. Quod ſi ergo ſcala relationis fuerit bimembris $\alpha, -\beta$; Seriei $A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Pz^n$, quæ oritur ex fractione $\frac{A + (B - \alpha A)z}{1 - \alpha z + \beta zz}$, ſumma erit

$$\frac{A + (B - \alpha A)z - Qz^{n+1} - (R - \alpha Q)z^{n+2}}{1 - \alpha z + \beta zz}.$$

At eſt, ex natura Seriei, $R = \alpha Q - \beta P$, unde prodibit ſumma

$$\frac{A + (B - \alpha A)z - Qz^{n+1} + \beta Pz^{n+2}}{1 - \alpha z + \beta zz}.$$

EXEMPLUM.

Sit propoſita Series $1 + 3z + 4z^2 + 7z^3 + \ldots\ldots + Pz^n$ ubi eſt $\alpha = 1$; $\beta = -1$; $A = 1$; $B = 3$; erit hujus ſumma $\frac{1 + 2z - Qz^{n+1} - Pz^{n+2}}{1 - z - zz}$. Poſito vero $z = 1$, erit ſumma Seriei $1 + 3 + 4 + 7 + 11 + \ldots\ldots\ldots + P = P + Q - 3$. Summa ergo termini ultimi & ſequentis ternario excedit ſummam Seriei. Quia vero eſt $Q = \frac{P + \sqrt{(5PP \pm 20)}}{2}$ erit ſumma Seriei $1 + 3 + 4 + 7 + 11 + \ldots + P = \frac{3P - 6 + \sqrt{(5PP \pm 20)}}{2}$. Ex ſolo ergo termino ultimo ſumma poteſt exhiberi.

CAPUT XIV.

De multiplicatione ac diviſione Angulorum.

234. Sit Angulus, vel Arcus, in Circulo cujus Radius $= 1$, quicunque $= z$, ejus Sinus $= x$, Coſinus $= y$ & Tangens $= t$; erit $xx + yy = 1$ & $t = \frac{x}{y}$. Cum igitur, uti ſupra vidimus, tam Sinus quam Coſinus Angulorum z; $2z$; $3z$; $4z$; $5z$; &c., conſtituant Seriem recurrentem cujus ſcala relationis eſt $2y, -1$; primum Sinus horum Arcuum ita ſe habebunt:

$$\begin{aligned}
ſin.\ 0z &= 0 \\
ſin.\ 1z &= x \\
ſin.\ 2z &= 2xy \\
ſin.\ 3z &= 4xy^2 - x \\
ſin.\ 4z &= 8xy^3 - 4xy \\
ſin.\ 5z &= 16xy^4 - 12xy^2 + x \\
ſin.\ 6z &= 32xy^5 - 32xy^3 + 6xy \\
ſin.\ 7z &= 64xy^6 - 80xy^4 + 24xy^2 - x \\
ſin.\ 8z &= 128xy^7 - 192xy^5 + 80xy^3 - 8xy
\end{aligned}$$

hinc concluditur fore

$$ſin.\ nz = x\,(2^{n-1}y^{n-1} - (n-2)\,2^{n-3}y^{n-3} + \frac{(n-3)(n-4)}{1.\ 2}\,2^{n-5}y^{n-5} - \frac{(n-4)(n-5)(n-6)}{1.\ 2.\ 3}\,2^{n-7}y^{n-7} + \frac{(n-5)(n-6)(n-7)(n-8)}{1.\ 2.\ 3.\ 4}\,2^{n-9}y^{n-9} - \&c.)$$

235. Si ponamus Arcum $nz = s$; erit $ſin.\ nz = ſin.\ s = ſin.\ (\pi - s) = ſin.\ (2\pi + s) = ſin.\ (3\pi - s)$ &c.,

hi enim Sinus omnes ſunt inter ſe æquales. Hinc obtinemus plures valores pro x, qui erunt

$$\text{ſin.}\,\frac{s}{n};\ \text{ſin.}\,\frac{\pi - s}{n};\ \text{ſin.}\,\frac{2\pi + s}{n};\ \text{ſin.}\,\frac{3\pi - s}{n};\ \text{ſin.}\,\frac{4\pi + s}{n};\ \&c.,$$

qui ergo omnes æquationi inventæ æque conveniunt. Tot autem prodibunt diverſi pro x valores, quot numerus n continet unitates, qui propterea erunt radices æquationis inventæ. Cavendum ergo eſt, ne valores æquales pro iiſdem habeantur, quod fiet dum alternæ tantum expreſſiones aſſumantur. Cognitis igitur radicibus æquationis a poſteriori, earum comparatio cum terminis æquationis notatu dignas præbebit proprietates. Quoniam autem ad hoc æquatio, in qua tantum x tanquam incognita inſit, requiritur, pro y ſuus valor $\sqrt{(1 - xx)}$ ſubſtitui debet; unde duplex operatio inſtituenda erit, prout n fuerit vel numerus par vel impar.

236. Sit n numerus impar, quia Arcuum $-z, +z, +3z, +5z$, &c., differentia eſt $2z$, hujuſque Coſinus $= 1 - 2xx$, erit progreſſionis Sinuum ſcala relationis hæc $2 - 4xx, - 1$. Hinc erit

$$\begin{aligned}
\text{ſin.} -z &= -x\\
\text{ſin.}\ z &= x\\
\text{ſin.}\ 3z &= 3x - 4x^3\\
\text{ſin.}\ 5z &= 5x - 20x^3 + 16x^5\\
\text{ſin.}\ 7z &= 7x - 56x^3 + 112x^5 - 64x^7\\
\text{ſin.}\ 9z &= 9x - 120x^3 + 432x^5 - 576x^7 + 256x^9
\end{aligned}$$

ergo

$$\text{ſin.}\ nz = nx - \frac{n(nn-1)}{1.\,2.\,3}x^3 + \frac{n(nn-1)(nn-9)}{1.\,2.\,3.\,4.\,5}x^5 - \frac{n(nn-1)(nn-9)(nn-25)}{1.\,2.\,3.\,4.\,5.\,6.\,7}x^7 + \&c.,$$

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LIB. I. ſi quidem n fuerit numerus impar. Hujuſque æquationis radices ſunt $\text{ſin.}\, z$; $\text{ſin.}\,(\frac{2\pi}{n}+z)$; $\text{ſin.}\,(\frac{4\pi}{n}+z)$; $\text{ſin.}\,(\frac{6\pi}{n}+z)$; $\text{ſin.}\,(\frac{8\pi}{n}+z)$; &c., quarum numerus eſt n.

237. Hujus ergo æquationis

$$0 = 1 - \frac{nx}{\text{ſin.}\, nz} + \frac{n(nn-1)x^3}{1.2.3\,\text{ſin.}\, nz} - \frac{n(nn-1)(nn-9)x^5}{1.2.3.4.5\,\text{ſin.}\, nz} \ldots\ldots \pm \frac{2^{n-1}x^n}{\text{ſin.}\, nz},$$

(ubi ſignum ſuperius valet ſi n unitate deficiat a multiplo quaternarii, contra inferius) Factores ſunt $(1-\frac{x}{\text{ſin.}\, z})$ $(1-\frac{x}{\text{ſin.}\,(\frac{2\pi}{n}+z)})(1-\frac{x}{\text{ſin.}\,(\frac{4\pi}{n}+z)})$ &c., ex quibus concluditur fore

$$\frac{n}{\text{ſin.}\, nz} = \frac{1}{\text{ſin.}\, z} + \frac{1}{\text{ſin.}\,(\frac{2\pi}{n}+z)} + \frac{1}{\text{ſin.}\,(\frac{4\pi}{n}+z)} + \frac{1}{\text{ſin.}\,(\frac{6\pi}{n}+z)} +$$

&c., donec habeantur n termini. Tum vero productum omnium erit $\mp \frac{2^{n-1}}{\text{ſin.}\, nz} = \frac{1}{\text{ſin.}\, z.\,\text{ſin.}\,(\frac{2\pi}{n}+z)\,\text{ſin.}\,(\frac{4\pi}{n}+z)\,\text{ſin.}\,(\frac{6\pi}{n}+z)\,\&c.}$ ſeu $\text{ſin.}\, nz = \mp 2^{n-1}\,\text{ſin.}\, z.\,\text{ſin.}\,(\frac{2\pi}{n}+z)\,\text{ſin.}\,(\frac{4\pi}{n}+z) \times \text{ſin.}\,(\frac{6\pi}{n}+z)$ &c.. Et, quia terminus penultimus deeſt, erit

$$0 = \text{ſin.}\, z + \text{ſin.}\,(\tfrac{2\pi}{n}+z) + \text{ſin.}\,(\tfrac{4\pi}{n}+z) + \text{ſin.}\,(\tfrac{6\pi}{n}+z)\ \&c.$$

EXEMPLUM I.

Si ergo fuerit $n=3$, prodibunt hæ æqualitates $0 = \text{ſin.}\, z + \text{ſin.}\,(120^\circ + z) + \text{ſin.}\,(240^\circ + z) = \text{ſin.}\, z + \text{ſin.}\,(60^\circ - z) - \text{ſin.}\,(60^\circ + z)$.

$\frac{3}{\text{ſin.}\, 3z}$

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$$\frac{3}{\text{ſin. } 3z} = \frac{1}{\text{ſin. } z} + \frac{1}{\text{ſin. } (120^\circ + z)} + \frac{1}{\text{ſin. } (240^\circ + z)} = \frac{1}{\text{ſin. } z} + \frac{1}{\text{ſin. } (60 - z)} - \frac{1}{\text{ſin. } (60 +}$$

$$\text{ſin. } 3z = -4\,\text{ſin. } z.\, \text{ſin. } (120^\circ + z)\, \text{ſin. } (240^\circ + z) = 4\,\text{ſin. } z.\, \text{ſin. } (60^\circ - z).\, \text{ſin. } (60^\circ + z).$$

Erit ergo, uti jam ſupra notavimus,

$$\text{ſin. } (60^\circ + z) = \text{ſin. } z + \text{ſin. } (60^\circ - z) \ \&$$
$$3\,\text{coſec. } 3z = \text{coſec. } z + \text{coſec. } (60^\circ - z) - \text{coſec. } (60^\circ + z).$$

EXEMPLUM II.

Ponamus eſſe $n = 5$, atque prodibunt hæ æquationes:

$$0 = \text{ſin. } z + \text{ſin. } (\tfrac{2}{5}\pi + z) + \text{ſin. } (\tfrac{4}{5}\pi + z) + \text{ſin. } (\tfrac{6}{5}\pi + z) + \text{ſin. } (\tfrac{8}{5}\pi + z)$$

ſeu $0 = \text{ſin. } z + \text{ſin. } (\tfrac{2}{5}\pi + z) + \text{ſin. } (\tfrac{1}{5}\pi - z) - \text{ſin. } (\tfrac{1}{5}\pi + z) - \text{ſin. } (\tfrac{2}{5}\pi - z)$

ſeu $0 = \text{ſin. } z + \text{ſin. } (\tfrac{1}{5}\pi - z) - \text{ſin. } (\tfrac{1}{5}\pi + z) + \text{ſin. } (\tfrac{2}{5}\pi + z) - \text{ſin. } (\tfrac{2}{5}\pi - z)$

deinde erit

$$\frac{5}{\text{ſin. } 5z} = \frac{1}{\text{ſin. } z} + \frac{1}{\text{ſin. } (\frac{1}{5}\pi - z)} - \frac{1}{\text{ſin. } (\frac{1}{5}\pi + z)} - \frac{1}{\text{ſin. } (\frac{2}{5}\pi - z)} + \frac{1}{\text{ſin. } (\frac{2}{5}\pi + z)}$$

$$\text{ſin. } 5z = 16\,\text{ſin. } z.\, \text{ſin. } (\tfrac{1}{5}\pi - z)\, \text{ſin. } (\tfrac{1}{5}\pi + z) \times \text{ſin. } (\tfrac{2}{5}\pi - z)\, \text{ſin. } (\tfrac{2}{5}\pi + z).$$

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LIB. I.

EXEMPLUM III.

Hoc modo, ſi ponamus $n = 2m + 1$, erit

$$0 = \textit{ſin.}\, z + \textit{ſin.}\,(\tfrac{\pi}{n} - z) - \textit{ſin.}\,(\tfrac{\pi}{n} + z) - \textit{ſin.}\,(\tfrac{2\pi}{n} - z) + \textit{ſin.}\,(\tfrac{2\pi}{n} + z) + \textit{ſin.}\,(\tfrac{3\pi}{n} - z) - \textit{ſin.}\,(\tfrac{3\pi}{n} + z) - \ldots \pm \textit{ſin.}\,(\tfrac{m}{n}\pi - z) \mp \textit{ſin.}\,(\tfrac{m}{n}\pi + z)$$

ubi ſigna ſuperiora valent ſi m ſit numerus impar, inferiora ſi ſit par. Altera æquatio erit hæc

$$\frac{n}{\textit{ſin.}\, nz} = \frac{1}{\textit{ſin.}\, z} + \frac{1}{\textit{ſin.}\,(\frac{\pi}{n} - z)} - \frac{1}{\textit{ſin.}\,(\frac{\pi}{n} + z)} - \frac{1}{\textit{ſin.}\,(\frac{2\pi}{n} - z)} + \frac{1}{\textit{ſin.}\,(\frac{2\pi}{n} + z)} + \frac{1}{\textit{ſin.}\,(\frac{3\pi}{n} - z)} - \frac{1}{\textit{ſin.}\,(\frac{3\pi}{n} + z)} - \ldots \pm \frac{1}{\textit{ſin.}\,(\frac{m\pi}{n} - z)} \mp \frac{1}{\textit{ſin.}\,(\frac{m\pi}{n} + z)}$$

quæ ad Coſécantes commode transfertur. Tertio habetur hoc productum:

$$\textit{ſin.}\, nz = 2^{2m}\, \textit{ſin.}\, z.\, \textit{ſin.}\,(\tfrac{\pi}{n} - z)\, \textit{ſin.}\,(\tfrac{\pi}{n} + z)\, \textit{ſin.}\,(\tfrac{2\pi}{n} - z) \times \textit{ſin.}\,(\tfrac{2\pi}{n} + z)\, \textit{ſin.}\,(\tfrac{3\pi}{n} - z)\, \textit{ſin.}\,(\tfrac{3\pi}{n} + z) \ldots\ldots \times \textit{ſin.}\,(\tfrac{m\pi}{n} - z)\, \textit{ſin.}\,(\tfrac{m\pi}{n} + z).$$

238. Sit n nunc numerus par, & quoniam eſt $y = \sqrt{(1 - xx)}$ & $\textit{coſ.}\, 2z = 1 - 2xx$, ita ut Seriei Sinuum ſit ſcala relationis, ut ante, $2 - 4xx, -1$, erit

ſin. $0z = 0$
ſin. $2z = 2x\sqrt{(1-xx)}$
ſin. $4z = (4x - 8x^3)\sqrt{(1-xx)}$
ſin. $6z = (6x - 32x^3 + 32x^5)\sqrt{(1-xx)}$
ſin. $8z = (8x - 80x^3 + 192x^5 - 128x^7)\sqrt{(1-xx)}$

& generaliter

ſin. $nz = \left(nx - \frac{n(nn-4)}{1.2.3}x^3 + \frac{n(nn-4)(nn-16)}{1.2.3.4.5}x^5 - \frac{n(nn-4)(nn-16)(nn-36)}{1.2.3.4.5.6.7}x^7 + \ldots \pm 2^{n-1}x^{n-1}\right)\sqrt{(1-xx)}$

denotante n numerum quemcunque parem.

239. Ad æquationem hanc rationalem efficiendam ſumantur utrinque quadrata, ac prodibit hujuſmodi æquatio

(ſin. $nz)^2 = nnxx + Px^4 + Qx^6 + \ldots - 2^{2n-2}x^{2n}$

ſeu $x^{2n} \ldots\ldots - \frac{nn}{2^{2n-2}}xx + \frac{1}{2^{2n-2}}$ (ſin. $nz)^2 = 0$

cujus æquationis radices erunt tam affirmativæ quam negativæ; Scilicet $\pm$ ſin. z; $\pm$ ſin. $(\frac{\pi}{n} - z)$; $\pm$ ſin. $(\frac{2\pi}{n} + z)$; $\pm$ ſin. $(\frac{3\pi}{n} - z)$; $\pm$ ſin. $(\frac{4\pi}{n} + z)$ &c. Sumendo omnino n hujuſmodi expreſſiones. Cum igitur ultimus terminus ſit productum omnium harum radicum, extrahendo utrinque radicem quadratam erit

ſin. $nz = \pm 2^{n-1}$ ſin. z. ſin. $(\frac{\pi}{n} - z)$ ſin. $(\frac{2\pi}{n} + z) \times$ ſin. $(\frac{3\pi}{n} - z) \ldots\ldots$; ubi, quibus caſibus utrumvis ſignum valeat, ex caſibus particularibus erit diſpiciendum.

EXEMPLUM.

Subſtituendo autem pro n ſucceſſive numeros 2, 4, 6, &c. & eligendo n Sinus diverſos erit.

$$\text{ſin.}\, 2z = 2\,\text{ſin.}\, z.\, \text{ſin.}\left(\frac{\pi}{2} - z\right)$$

$$\text{ſin.}\, 4z = 8\,\text{ſin.}\, z.\, \text{ſin.}\left(\frac{\pi}{4} - z\right) \text{ſin.}\left(\frac{\pi}{4} + z\right) \text{ſin.}\left(\frac{\pi}{2} - z\right)$$

$$\text{ſin.}\, 6z = 32\,\text{ſin.}\, z.\, \text{ſin.}\left(\frac{\pi}{6} - z\right) \text{ſin.}\left(\frac{\pi}{6} + z\right) \text{ſin.}\left(\frac{2\pi}{6} - z\right) \times \text{ſin.}\left(\frac{2\pi}{6} + z\right) \text{ſin.}\left(\frac{3\pi}{6} - z\right)$$

$$\text{ſin.}\, 8z = 128\,\text{ſin.}\, z.\, \text{ſin.}\left(\frac{\pi}{8} - z\right) \text{ſin.}\left(\frac{\pi}{8} + z\right) \text{ſin.}\left(\frac{2\pi}{8} - z\right) \times \text{ſin.}\left(\frac{2\pi}{8} + z\right) \text{ſin.}\left(\frac{3\pi}{8} - z\right) \text{ſin.}\left(\frac{3\pi}{8} + z\right) \text{ſin.}\left(\frac{4\pi}{8} - z\right).$$

240. Patet ergo fore generatim

$$\text{ſin.}\, nz = 2^{n-1}\,\text{ſin.}\, z.\, \text{ſin.}\left(\frac{\pi}{n} - z\right) \text{ſin.}\left(\frac{\pi}{n} + z\right) \text{ſin.}\left(\frac{2\pi}{n} - z\right) \times \text{ſin.}\left(\frac{2\pi}{n} + z\right) \text{ſin.}\left(\frac{3\pi}{n} - z\right) \text{ſin.}\left(\frac{3\pi}{n} + z\right) \ldots \text{ſin.}\left(\frac{1}{2}\pi - z\right)$$

ſi n fuerit numerus par. Quod ſi autem hæc cum ſuperiori, ubi n erat numerus impar, comparetur, tanta ſimilitudo adeſſe deprehenditur, ut utramque in unam redigere liceat. Erit ergo, ſive n fuerit numerus par ſive impar,

$$\text{ſin.}\, nz = 2^{n-1}\,\text{ſin.}\, z.\, \text{ſin.}\left(\frac{\pi}{n} - z\right). \text{ſin.}\left(\frac{\pi}{n} + z\right) \text{ſin.}\left(\frac{2\pi}{n} - z\right) \times \text{ſin.}\left(\frac{2\pi}{n} + z\right) \text{ſin.}\left(\frac{3\pi}{n} - z\right) \text{ſin.}\left(\frac{3\pi}{n} + z\right) \&c.$$

donec tot habeantur Factores, quot numerus n continet unitates.

241. Expreſſiones iſtæ, quibus Sinus Angulorum multiplorum per Factores exponuntur, non parum utilitatis afferre poſſunt ad Logarithmos Sinuum Angulorum multiplorum inveniendos, itemque ad plures expreſſiones Sinuum per Factores, quales ſupra (§. 184.) dedimus, reperiendas. Erit autem

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$$\text{ſin.}\ z = 1\,\text{ſin.}\ z$$
$$\text{ſin.}\ 2z = 2\,\text{ſin.}\ z.\,\text{ſin.}\left(\frac{\pi}{2} - z\right)$$
$$\text{ſin.}\ 3z = 4\,\text{ſin.}\ z.\,\text{ſin.}\left(\frac{\pi}{3} - z\right)\text{ſin.}\left(\frac{\pi}{3} + z\right)$$
$$\text{ſin.}\ 4z = 8\,\text{ſin.}\ z.\,\text{ſin.}\left(\frac{\pi}{4} - z\right)\text{ſin.}\left(\frac{\pi}{4} + z\right)\text{ſin.}\left(\frac{2\pi}{4} - z\right)$$
$$\text{ſin.}\ 5z = 16\,\text{ſin.}\ z.\,\text{ſin.}\left(\frac{\pi}{5} - z\right)\text{ſin.}\left(\frac{\pi}{5} + z\right)\text{ſin.}\left(\frac{2\pi}{5} - z\right)\times \text{ſin.}\left(\frac{2\pi}{5} + z\right)$$
$$\text{ſin.}\ 6z = 32\,\text{ſin.}\ z.\,\text{ſin.}\left(\frac{\pi}{6} - z\right)\text{ſin.}\left(\frac{\pi}{6} + z\right)\text{ſin.}\left(\frac{2\pi}{6} - z\right)\times \text{ſin.}\left(\frac{2\pi}{6} + z\right)\text{ſin.}\left(\frac{3\pi}{6} - z\right)$$

&c.

242. Cum deinde ſit $\frac{\text{ſin.}\ 2nz}{\text{ſin.}\ nz} = 2\,\text{coſ.}\ nz$, Coſinus Angulorum multiplorum ſimili modo per Factores exprimentur:

$$\text{coſ.}\ z = 1\,\text{ſin.}\left(\frac{\pi}{2} - z\right)$$
$$\text{coſ.}\ 2z = 2\,\text{ſin.}\left(\frac{\pi}{4} - z\right)\text{ſin.}\left(\frac{\pi}{4} + z\right)$$
$$\text{coſ.}\ 3z = 4\,\text{ſin.}\left(\frac{\pi}{6} - z\right)\text{ſin.}\left(\frac{\pi}{6} + z\right)\text{ſin.}\left(\frac{3\pi}{6} - z\right)$$
$$\text{coſ.}\ 4z = 8\,\text{ſin.}\left(\frac{\pi}{8} - z\right)\text{ſin.}\left(\frac{\pi}{8} + z\right)\text{ſin.}\left(\frac{3\pi}{8} - z\right)\times \text{ſin.}\left(\frac{3\pi}{8} + z\right)$$
$$\text{coſ.}\ 5z = 16\,\text{ſin.}\left(\frac{\pi}{10} - z\right)\text{ſin.}\left(\frac{\pi}{10} + z\right)\text{ſin.}\left(\frac{3\pi}{10} - z\right)\times \text{ſin.}\left(\frac{3\pi}{10} + z\right)\text{ſin.}\left(\frac{5\pi}{10} - z\right)$$

& generaliter

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LIB. I. $cof. nz = 2^{n-1} ſin.(\frac{\pi}{2n} - z)\, ſin.(\frac{\pi}{2n} + z)\, ſin.(\frac{3\pi}{2n} - z) \times$

$$ſin.(\frac{3\pi}{2n} + z)\, ſin.(\frac{5\pi}{2n} - z) \text{ \&c.},$$

quoad tot habeantur Factores quot numerus n continet unitates.

243. Eædem expreſſiones prodibunt ex conſideratione Coſinuum Arcuum multiplorum; ſi enim fuerit $coſ. z = y$, erit ut ſequitur

$$\begin{aligned}
coſ.\, 0z &= 1 \\
coſ.\, 1z &= y \\
coſ.\, 2z &= 2yy - 1 \\
coſ.\, 3z &= 4y^3 - 3y \\
coſ.\, 4z &= 8y^4 - 8yy + 1 \\
coſ.\, 5z &= 16y^5 - 20y^3 + 5y \\
coſ.\, 6z &= 32y^6 - 48y^4 + 18yy - 1 \\
coſ.\, 7z &= 64y^7 - 112y^5 + 56y^3 - 7y
\end{aligned}$$

& generaliter

$$coſ.\, nz = 2^{n-1} y^n - \frac{n}{1} 2^{n-3} y^{n-2} + \frac{n(n-3)}{1.\ 2} \times 2^{n-5} y^{n-4} - \frac{n(n-4)(n-5)}{1.\ 2.\ 3} 2^{n-7} y^{n-6} + \frac{n(n-5)(n-6)(n-7)}{1.\ 2.\ 3.\ 4} 2^{n-9} y^{n-8} - \text{\&c.},$$

cujus æquationis, cum ſit $coſ.\, nz = coſ.(2\pi - nz) = coſ.(2\pi + nz) = coſ.(4\pi \pm nz) = coſ.(6\pi \pm nz)$ &c., erunt radices ipſius y hæ: $coſ.\, z$; $coſ.(\frac{2\pi}{n} \pm z)$; $coſ.(\frac{4\pi}{n} \pm z)$; $coſ.(\frac{6\pi}{n} \pm z)$ &c., quarum formularum tot diverſæ ſunt pro y eligendæ quot dantur; dantur autem tot, quot n continet unitates.

244. Primum igitur patet, ob terminum ſecundum deficientem excepto caſu $n = 1$, fore ſummam harum radicum omnium $= 0$. Erit ergo CAP. XIV.

$$0 = cof.z + cof.(\tfrac{2\pi}{n} - z) + cof.(\tfrac{2\pi}{n} + z) + cof.(\tfrac{4\pi}{n} - z) + cof.(\tfrac{4\pi}{n} + z) + \&c.,$$

ſumendo tot terminos quot n continet unitates: hæc autem æqualitas ſponte ſe offert ſi n ſit numerus par, cum quivis terminus ab alio ſui negativo deſtruatur. Contemplemur ergo numeros impares, unitate excluſa, eritque, ob $cof.\,v = - cof.\,(\pi - v)$,

$$0 = cof.z - cof.(\tfrac{\pi}{3} - z) - cof.(\tfrac{\pi}{3} + z)$$

$$0 = cof.z - cof.(\tfrac{\pi}{5} - z) - cof.(\tfrac{\pi}{5} + z) + cof.(\tfrac{2\pi}{5} - z) + cof.(\tfrac{2\pi}{5} + z)$$

$$0 = cof.z - cof.(\tfrac{\pi}{7} - z) - cof.(\tfrac{\pi}{7} + z) + cof.(\tfrac{2\pi}{7} - z) + cof.(\tfrac{2\pi}{7} + z) - cof.(\tfrac{3\pi}{7} - z) - cof.(\tfrac{3\pi}{7} + z)$$

& generaliter, ſi fuerit n numerus impar quicunque, erit

$$0 = cof.z - cof.(\tfrac{\pi}{n} - z) - cof.(\tfrac{\pi}{n} + z) + cof.(\tfrac{2\pi}{n} - z) + cof.(\tfrac{2\pi}{n} + z) - cof.(\tfrac{3\pi}{n} - z) - cof.(\tfrac{3\pi}{n} + z) + cof.(\tfrac{4\pi}{n} - z) + cof.(\tfrac{4\pi}{n} + z) - \&c.,$$

ſumendo tot terminos, quot numerus n continet unitates: oportet autem n eſſe numerum imparem unitate majorem, uti jam monuimus.

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LIB. I. 245. Quod ad productum ex omnibus attinet, variæ quidem prodeunt expreſſiones, prout n fuerit numerus vel impar, vel impariter par, vel pariter par: omnes autem comprehenduntur in expreſſione generali (§. 242) inventa, ſi ſinguli Sinus in Coſinus tranſmutentur: erit ſcilicet

$$cof.\ z = 1\ cof.\ z$$
$$cof.\ 2z = 2\ cof.\left(\tfrac{\pi}{4}+z\right) cof.\left(\tfrac{\pi}{4}-z\right)$$
$$cof.\ 3z = 4\ cof.\left(\tfrac{2\pi}{6}+z\right) cof.\left(\tfrac{2\pi}{6}-z\right) cof.\ z$$
$$cof.\ 4z = 8\ cof.\left(\tfrac{3\pi}{8}+z\right) cof.\left(\tfrac{3\pi}{8}-z\right) cof.\left(\tfrac{\pi}{8}+z\right)\times cof.\left(\tfrac{\pi}{8}-z\right)$$
$$cof.\ 5z = 16\ cof.\left(\tfrac{4\pi}{10}+z\right) cof.\left(\tfrac{4\pi}{10}-z\right) cof.\left(\tfrac{2\pi}{10}+z\right)\times cof.\left(\tfrac{2\pi}{10}-z\right) cof.\ z;$$

& generaliter

$$cof.\ nz = 2^{n-1}\ cof.\left(\tfrac{n-1}{2n}\pi+z\right) cof.\left(\tfrac{n-1}{2n}\pi-z\right)\times cof.\left(\tfrac{n-3}{2n}\pi+z\right) cof.\left(\tfrac{n-3}{2n}\pi-z\right)\times cof.\left(\tfrac{n-5}{2n}\pi+z\right) cof.\left(\tfrac{n-5}{2n}\pi-z\right)\times cof.\left(\tfrac{n-7}{2n}\pi+z\right) \&c.,$$

ſumtis tot Factoribus, quot numerus n continet unitates.

246. Sit n numerus impar, atque æquatio incipiatur ab unitate, erit $0 = 1 \mp \frac{ny}{cof.\ nz} + \&c.$, ubi ſignum ſuperius valet ſi n fuerit numerus impar formæ $4m+1$, inferius ſi $n = 4m-1$. Hinc erit

+

$$+\frac{1}{\cos z}=\frac{1}{\cos z}$$

$$-\frac{3}{\cos 3z}=\frac{1}{\cos z}-\frac{1}{\cos\left(\frac{\pi}{3}-z\right)}-\frac{1}{\cos\left(\frac{\pi}{3}+z\right)}$$

$$+\frac{5}{\cos 5z}=\frac{1}{\cos z}-\frac{1}{\cos\left(\frac{\pi}{5}-z\right)}-\frac{1}{\cos\left(\frac{\pi}{5}+z\right)}+\frac{1}{\cos\left(\frac{2\pi}{5}-z\right)}+\frac{1}{\cos\left(\frac{2\pi}{5}+z\right)}$$

& generaliter, poſito $n=2m+1$, erit

$$\frac{n}{\cos nz}=\frac{2m+1}{\cos(2m+1)z}=\frac{1}{\cos\left(\frac{m}{n}\pi+z\right)}+\frac{1}{\cos\left(\frac{m}{n}\pi-z\right)}-\frac{1}{\cos\left(\frac{m-1}{n}\pi+z\right)}-\frac{1}{\cos\left(\frac{m-1}{n}\pi-z\right)}+\frac{1}{\cos\left(\frac{m-2}{n}\pi+z\right)}+\frac{1}{\cos\left(\frac{m-2}{n}\pi-z\right)}-\frac{1}{\cos\left(\frac{m-3}{n}\pi+z\right)}-\&c.$$

ſumendis tot terminis, quot n continet unitates.

247. Cum ergo ſit $\frac{1}{\cos v}=\sec v$, hinc pro Secantibus inſignes proprietates deducantur, erit nempe

$$\sec z=\sec z$$

$$3\sec 3z=\sec\left(\frac{\pi}{3}+z\right)+\sec\left(\frac{\pi}{3}-z\right)-\sec\left(\frac{0\pi}{3}+z\right)$$

$$5\sec 5z=\sec\left(\frac{2\pi}{5}+z\right)+\sec\left(\frac{2\pi}{5}-z\right)-\sec\left(\frac{\pi}{5}+z\right)-\sec\left(\frac{\pi}{5}-z\right)+\sec\left(\frac{0\pi}{5}+z\right)$$

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$$7\,ſec.\,7z = ſec.\left(\frac{3\pi}{7}+z\right) + ſec.\left(\frac{3\pi}{7}-z\right) - ſec.\left(\frac{2\pi}{7}+z\right) - ſec.\left(\frac{2\pi}{7}-z\right) - ſec.\left(\frac{\pi}{7}+z\right) + ſec.\left(\frac{\pi}{7}-z\right) - ſec.\left(\frac{0\pi}{7}+z\right)$$

& generaliter, poſito $n = 2m + 1$, erit

$$n.\,ſec.\,nz = ſec.\left(-\frac{m}{n}\pi + z\right) + ſec.\left(-\frac{m}{n}\pi - z\right) - ſec.\left(\frac{m-1}{n}\pi + z\right) - ſec.\left(\frac{m-1}{n}\pi - z\right) + ſec.\left(\frac{m-2}{n}\pi + z\right) + ſec.\left(\frac{m-2}{n}\pi - z\right) - ſec.\left(\frac{m-3}{n}\pi + z\right) - ſec.\left(\frac{m-3}{n}\pi - z\right) + ſec.\left(\frac{m-4}{n}\pi + z\right) + \ldots\ldots\ldots \pm ſec.\,z.$$

248. Pro Coſecantibus autem erit ex §. 237:

$$coſec.\ z = coſec.\,z$$

$$3\,coſec.\,3z = coſec.\,z + coſec.\left(\frac{\pi}{3}-z\right) - coſec.\left(\frac{\pi}{3}+z\right)$$

$$5\,coſec.\,5z = coſec.\,z + coſec.\left(\frac{\pi}{5}-z\right) - coſec.\left(\frac{\pi}{5}+z\right) - coſec.\left(\frac{2\pi}{5}-z\right) + coſec.\left(\frac{2\pi}{5}+z\right)$$

$$7\,coſec.\,7z = coſec.\,z + coſec.\left(\frac{\pi}{7}-z\right) - coſec.\left(\frac{\pi}{7}+z\right) - coſec.\left(\frac{2\pi}{7}-z\right) + coſec.\left(\frac{2\pi}{7}+z\right) + coſec.\left(\frac{3\pi}{7}-z\right) - coſec.\left(\frac{3\pi}{7}+z\right)$$

& generaliter, ponendo $n = 2m + 1$, erit

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$$n.\,coſec.\,n z = coſec.\,z + coſec.\left(\frac{\pi}{n} - z\right) - coſec.\left(\frac{\pi}{n} + z\right) - coſec.\left(\frac{2\pi}{n} - z\right) + coſec.\left(\frac{2\pi}{n} + z\right) + coſec.\left(\frac{3\pi}{n} - z\right) - coſec.\left(\frac{3\pi}{n} + z\right) - \ldots\ldots\ldots\ldots \mp coſec.\left(\frac{m\pi}{n} - z\right) \pm coſec.\left(\frac{m\pi}{n} + z\right)$$

ubi ſigna ſuperiora valent ſi m fuerit numerus par, inferiora ſi m ſit impar.

249. Cum ſit, uti ſupra vidimus, $coſ.\,nz \pm \sqrt{-1}.\,ſin.\,nz = (coſ.\,z \pm \sqrt{-1}.\,ſin.\,z)^n$, erit $coſ.\,nz = \frac{(coſ.\,z + \sqrt{-1}.\,ſin.\,z)^n + (coſ.\,z - \sqrt{-1}.\,ſin.\,z)^n}{2}$; $ſin.\,nz = \frac{(coſ.\,z + \sqrt{-1}.\,ſin.\,z)^n - (coſ.\,z - \sqrt{-1}.\,ſin.\,z)^n}{2\sqrt{-1}}$ &

$$tang.\,nz = \frac{(coſ.\,z + \sqrt{-1}.\,ſin.\,z)^n - (coſ.\,z - \sqrt{-1}.\,ſin.\,z)^n}{(coſ.\,z + \sqrt{-1}.\,ſin.\,z)^n\sqrt{-1} + (coſ.\,z - \sqrt{-1}.\,ſin.\,z)^n\sqrt{-1}}.$$

Ponamus $tang.\,z = \frac{ſin.\,z}{coſ.\,z} = t$, erit $tang.\,nz = \frac{(1 + t\sqrt{-1})^n - (1 - t\sqrt{-1})^n}{(1 + t\sqrt{-1})^n\sqrt{-1} + (1 - t\sqrt{-1})^n\sqrt{-1}}$, unde oriuntur Tangentes Angulorum multiplorum ſequentes

$$tang.\ z = t$$
$$tang.\ 2z = \frac{2t}{1 - tt}$$
$$tang.\ 3z = \frac{3t - t^3}{1 - 3tt}$$
$$tang.\ 4z = \frac{4t - 4t^3}{1 - 6tt + t^4}$$
$$tang.\ 5z = \frac{5t - 10t^3 + t^5}{1 - 10tt + 5t^4}$$

& generaliter

$$tang.nz = \frac{nt - \frac{n(n-1)(n-2)}{1.\ 2.\ 3}t^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ 3.\ 4.\ 5}t^5 - \&c.}{1 - \frac{n(n-1)}{1.\ 2}tt + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}t^4 - \&c.}.$$

Cum jam ſit $tang.nz = tang.(\pi + nz) = tang.(2\pi + nz) = tang.(3\pi + nz)$ &c., erunt valores ipſius t, ſeu radices æquationis, hæ, $tang.z$; $tang.(\frac{\pi}{n} + z)$; $tang.(\frac{2\pi}{n} + z)$; $tang.(\frac{3\pi}{n} + z)$; &c., quarum numerus eſt n.

250. Quod ſi æquatio ab unitate incipiat, erit

$$0 = 1 - \frac{nt}{tang.\ nz} - \frac{n(n-1)tt}{1.\ 2} + \frac{n(n-1)(n-2)t^3}{1.\ 2.\ 3\ tang.\ nz} + \&c..$$

Ex comparatione ergo coëfficientium cum radicibus, erit

$$n.cot.nz = cot.z + cot.(\frac{\pi}{n} + z) + cot.(\frac{2\pi}{n} + z) + cot.(\frac{3\pi}{n} + z) + cot.(\frac{4\pi}{n} + z) + \dots\dots + cot.(\frac{n-1}{n}\pi + z)$$

deinde erit ſumma quadratorum harum Cotangentium omnium $= \frac{nn}{(ſin.\ nz)^2} - n$, ſimilique modo ulteriores Poteſtates poſſunt definiri. Ponendo autem loco n numeros definitos, erit.

$$cot.\ z = cot.\ z$$

$$2\,cot.\ 2z = cot.\ z + cot.(\frac{\pi}{2} + z)$$

$$3\,cot.\ 3z = cot.\ z + cot.(\frac{\pi}{3} + z) + cot.(\frac{2\pi}{3} + z)$$

$$4\,cot.\ 4z = cot.\ z + cot.(\frac{\pi}{4} + z) + cot.(\frac{2\pi}{4} + z) + cot.(\frac{3\pi}{4} + z)$$

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$$5\cot. 5z = \cot. z + \cot.(\tfrac{\pi}{5}+z) + \cot.(\tfrac{2\pi}{5}+z) + \cot.(\tfrac{3\pi}{5}+z) + \cot.(\tfrac{4\pi}{5}+z).$$ CAP. XIV.

251. Quia vero eſt $\cot. v = -\cot.(\pi - v)$, erit

$$\cot. z = \cot. z$$
$$2\cot. 2z = \cot. z - \cot.(\tfrac{\pi}{2}-z)$$
$$3\cot. 3z = \cot. z - \cot.(\tfrac{\pi}{3}-z) + \cot.(\tfrac{\pi}{3}+z)$$
$$4\cot. 4z = \cot. z - \cot.(\tfrac{\pi}{4}-z) + \cot.(\tfrac{\pi}{4}+z) - \cot.(\tfrac{2\pi}{4}-z)$$
$$5\cot. 5z = \cot. z - \cot.(\tfrac{\pi}{5}-z) + \cot.(\tfrac{\pi}{5}+z) - \cot.(\tfrac{2\pi}{5}-z) + \cot.(\tfrac{2\pi}{5}+z)$$

& generaliter

$$n.\cot. nz = \cot. z - \cot.(\tfrac{\pi}{n}-z) + \cot.(\tfrac{\pi}{n}+z) - \cot.(\tfrac{2\pi}{n}-z) + \cot.(\tfrac{2\pi}{n}+z) - \cot.(\tfrac{3\pi}{n}-z) + \cot.(\tfrac{3\pi}{n}+z) - \&c.$$

donec tot habeantur termini, quot numerus n continet unitates.

252. Incipiamus æquationem inventam a Poteſtate ſumma, ubi primum diſtinguendi ſunt caſus, quibus n eſt vel numerus par, vel impar. Sit n numerus impar, ſeu $n = 2m + 1$, erit

LIB. I. $t - tang.\, z = 0$

$$t^3 - 3tt.\,tang.\,3z - 3t + tang.\,3z = 0$$

$$t^5 - 5t^4.\,tang.\,5z - 10t^3 + 10tt.\,tang.\,5z + 5t - tang.\,5z = 0$$

& generaliter

$$t^n - nt^{n-1}.\,tang.\,nz - \ldots\ldots\ldots\ldots\ldots \mp tang.\,nz = 0$$

ubi ſignum ſuperius — valet, ſi m ſit numerus par, inferius + ſi m ſit numerus impar. Erit ergo ex coëfficiente ſecundi termini

$$tang.\; z = tang.\, z$$

$$3\, tang.\, 3z = tang.\, z + tang.\left(\frac{\pi}{3} + z\right) + tang.\left(\frac{2\pi}{3} + z\right)$$

$$5\, tang.\, 5z = tang.\, z + tang.\left(\frac{\pi}{5} + z\right) + tang.\left(\frac{2\pi}{5} + z\right) + tang.\left(\frac{3\pi}{5} + z\right) + tang.\left(\frac{4\pi}{5} + z\right).$$

&c.

253. Cum igitur ſit $tang.\, v = -tang.\,(\pi - v)$, Anguli recto majores ad Angulos recto minores reducuntur, eritque

$$tang.\; z = tang.\, z$$

$$3\, tang.\, 3z = tang.\, z - tang.\left(\frac{\pi}{3} - z\right) + tang.\left(\frac{\pi}{3} + z\right)$$

$$5\, tang.\, 5z = tang.\, z - tang.\left(\frac{\pi}{5} - z\right) + tang.\left(\frac{\pi}{5} + z\right) - tang.\left(\frac{2\pi}{5} - z\right) + tang.\left(\frac{2\pi}{5} + z\right)$$

$$7\, tang.\, 7z = tang.\, z - tang.\left(\frac{\pi}{7} - z\right) + tang.\left(\frac{\pi}{7} + z\right) - tang.\left(\frac{2\pi}{7} - z\right) + tang.\left(\frac{2\pi}{7} + z\right) - tang.\left(\frac{3\pi}{7} - z\right) + tang.\left(\frac{3\pi}{7} + z\right)$$

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& generaliter, ſi $n = 2m + 1$, erit

$$n.\, tang.\, nz = tang.\, z - tang.\, \left(\frac{\pi}{n} - z\right) + tang.\, \left(\frac{\pi}{n} + z\right) - tang.\, \left(\frac{2\pi}{n} - z\right) + tang.\, \left(\frac{2\pi}{n} + z\right) - tang.\, \left(\frac{3\pi}{n} - z\right) + \ldots\ldots\ldots - tang.\, \left(\frac{m\pi}{n} - z\right) + tang.\, \left(\frac{m\pi}{n} + z\right).$$

254. Tum vero productum ex his Tangentibus omnibus erit $= tang.\, nz$, propterea quod per ſignorum negativorum numerum alternatim parem & imparem, ſuperior ſignorum ambiguitas tollitur. Sic erit

$$tang.\, z = tang.\, z$$

$$tang.\, 3z = tang.\, z.\, tang.\left(\frac{\pi}{3} - z\right).\, tang.\left(\frac{\pi}{3} + z\right)$$

$$tang.\, 5z = tang.\, z.\, tang.\left(\frac{\pi}{5} - z\right).\, tang.\left(\frac{\pi}{5} + z\right).\, tang.\left(\frac{2\pi}{5} - z\right) \times tang.\left(\frac{2\pi}{5} + z\right)$$

& generaliter, ſi $n = 2m + 1$, erit

$$tang.\, nz = tang.\, z.\, tang.\left(\frac{\pi}{n} - z\right).\, tang.\left(\frac{\pi}{n} + z\right).\, tang.\left(\frac{2\pi}{n} - z\right) \times tang.\left(\frac{2\pi}{n} + z\right).\, tang.\left(\frac{3\pi}{n} - z\right) \ldots\ldots\ldots \times tang.\left(\frac{m\pi}{n} - z\right).\, tang.\left(\frac{m\pi}{n} + z\right).$$

255. Sit jam n numerus par, atque, incipiendo a Poteſtate ſumma, erit

$$tt + 2t.\, cot.\, 2z - 1 = 0$$
$$t^4 + 4t^3.\, cot.\, 4z - 6tt - 4t.\, cot.\, 4z + 1 = 0$$

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& generaliter, ſi $n = 2m$, erit

$$t^n + nt^{n-1}.\,cot.\,n z - \ldots\ldots\ldots\ldots\ldots \mp 1 = 0$$

ubi ſignum ſuperius — valet ſi m ſit numerus impar, inferius + ſi m ſit par. Comparando ergo radices cum coëfficiente ſecundi termini, erit

$$-2\,cot.\,2z = tang.\,z + tang.\left(\tfrac{\pi}{2} + z\right)$$

$$-4\,cot.\,4z = tang.\,z + tang.\left(\tfrac{\pi}{4} + z\right) + tang.\left(\tfrac{2\pi}{4} + z\right) + tang.\left(\tfrac{3\pi}{4} + z\right)$$

$$-6\,cot.\,6z = tang.\,z + tang.\left(\tfrac{\pi}{6} + z\right) + tang.\left(\tfrac{2\pi}{6} + z\right) + tang.\left(\tfrac{3\pi}{6} + z\right) + tang.\left(\tfrac{4\pi}{6} + z\right) + tang.\left(\tfrac{5\pi}{6} + z\right).$$

&c.

256. Cum ſit $tang.\,v = -\,tang.\,(\pi - v)$, ſequentes formabuntur æquationes

$$2\,cot.\,2z = -\,tang.\,z + tang.\left(\tfrac{\pi}{2} - z\right)$$

$$4\,cot.\,4z = -\,tang.\,z + tang.\left(\tfrac{\pi}{4} - z\right) - tang.\left(\tfrac{\pi}{4} + z\right) + tang.\left(\tfrac{2\pi}{4} - z\right)$$

$$6\,cot.\,6z = -\,tang.\,z + tang.\left(\tfrac{\pi}{6} - z\right) - tang.\left(\tfrac{\pi}{6} + z\right) + tang.\left(\tfrac{2\pi}{6} - z\right) - tang.\left(\tfrac{2\pi}{6} + z\right) + tang.\left(\tfrac{3\pi}{6} - z\right).$$

&

& generaliter, ſi $n = 2m$, erit

$$n.\,cot.\,nz = -\,tang.\,z + tang.\left(\frac{\pi}{n} - z\right) - tang.\left(\frac{\pi}{n} + z\right) +$$
$$tang.\left(\frac{2\pi}{n} - z\right) - tang.\left(\frac{2\pi}{n} + z\right) +$$
$$tang.\left(\frac{3\pi}{n} - z\right) - tang.\left(\frac{3\pi}{n} + z\right) +$$
$$\ldots\ldots\ldots\ldots + tang.\left(\frac{m\pi}{n} - z\right).$$

257. Per has formas iterum ambiguitas producti ex omnibus radicibus deſtruitur; eritque idcirco

$$1 = tang.\,z.\,tang.\left(\frac{\pi}{2} - z\right)$$
$$1 = tang.\,z.\,tang.\left(\frac{\pi}{4} - z\right)\,tang.\left(\frac{\pi}{4} + z\right)\,tang.\left(\frac{2\pi}{4} - z\right)$$
$$1 = tang.\,z.\,tang.\left(\frac{\pi}{6} - z\right)\,tang.\left(\frac{\pi}{6} + z\right)\,tang.\left(\frac{2\pi}{6} - z\right)\times$$
$$tang.\left(\frac{2\pi}{6} + z\right)\,tang.\left(\frac{3\pi}{6} - z\right)$$
&c.

Harum vero æquationum ratio ſtatim ſponte in oculos incurrit, cum perpetuo bini Anguli reperiantur, quorum alter eſt alterius complementum ad rectum. Hujuſmodi ergo binorum Angulorum Tangentes productum dant $= 1$; ideoque omnium productum unitati debet eſſe æquale.

258. Quoniam Sinus & Coſinus Angulorum progreſſionem arithmeticam conſtituentium Seriem [illegible]rrentem præbent, per Caput præcedens ſumma hujuſmodi [illegible]uum & Coſ[illegible]uum quotcunque exhiberi poterit. Sint Ang[illegible]ithmetica progreſſione

$$a,\; a+b,\; a+2b,\; a+3b,\; a+4b,\; a+5b,\; \&c.$$

& quæratur primo ſumma Sinuum horum Angulorum in infinitum progredientium; ponatur ergo

$s = \text{ſin}.\,a + \text{ſin}.\,(a+b) + \text{ſin}.\,(a+2b) + \text{ſin}.\,(a+3b) + \&c.$
& quia hæc Series eſt recurrens, cujus ſcala relationis eſt $2\,\text{coſ}.\,b, -1$, orietur hæc Series ex evolutione fractionis, cujus denominator eſt $1 - 2z.\,\text{coſ}.\,b + zz$, poſito $z = 1$. Ipſa vero fractio erit $= \frac{\text{ſin}.\,a + z(\text{ſin}.\,(a+b) - 2\,\text{ſin}.\,a.\,\text{coſ}.\,b)}{1 - 2z.\,\text{coſ}.\,b + zz}$, quare, facto $z = 1$, erit $s = \frac{\text{ſin}.\,a + \text{ſin}.\,(a+b) - 2\,\text{ſin}.\,a.\,\text{coſ}.\,b}{2 - 2\,\text{coſ}.\,b} = \frac{\text{ſin}.\,a - \text{ſin}.\,(a-b)}{2(1 - \text{coſ}.\,b)}$, ob $2\,\text{ſin}.\,a.\,\text{coſ}.\,b = \text{ſin}.\,(a+b) + \text{ſin}.\,(a-b)$. Cum autem ſit $\text{ſin}.\,f - \text{ſin}.\,g = 2\,\text{coſ}.\,\frac{f+g}{2}.\,\text{ſin}.\,\frac{f-g}{2}$, erit $\text{ſin}.\,a - \text{ſin}.\,(a-b) = 2\,\text{coſ}.\,(a - \frac{1}{2}b)\,\text{ſin}.\,\frac{1}{2}b$: & $1 - \text{coſ}.\,b = 2(\text{ſin}.\,\frac{1}{2}b)^2$, unde erit $s = \frac{\text{coſ}.\,(a - \frac{1}{2}b)}{2\,\text{ſin}.\,\frac{1}{2}b}$.

259. Hinc itaque ſumma quotcunque Sinuum, quorum Arcus in arithmetica progreſſione incedunt, aſſignari poterit; quæratur nempe ſumma hujus progreſſionis
$\text{ſin}.\,a + \text{ſin}.\,(a+b) + \text{ſin}.\,(a+2b) + \text{ſin}.\,(a+3b) + \ldots\ldots\ldots + \text{ſin}.\,(a+nb)$.
Quia ſumma hujus progreſſionis in infinitum continuatæ eſt $\frac{\text{coſ}.\,(a - \frac{1}{2}b)}{2\,\text{ſin}.\,\frac{1}{2}b}$, conſiderentur termini ultimum ſequentes in infinitum hi
$\text{ſin}.\,(a+(n+1)b) + \text{ſin}.\,(a+(n+2)b) + \text{ſin}.\,(a+(n+3)b) + \&c.$
quia horum Sinuum ſumma eſt $= \frac{\text{coſ}.\,(a+(n+\frac{1}{2})b)}{2\,\text{ſin}.\,\frac{1}{2}b}$, ſi hæc a priori ſubtrahatur, remanebit ſumma quæſita. Scilicet, ſi fuerit
$s = \text{ſin}.\,a + \text{ſin}.\,(a+b) + \text{ſin}.\,(a+2b) + \ldots\ldots\ldots\ldots + \text{ſin}.\,(a+nb)$,
erit $s = \frac{\text{coſ}.\,(a - \frac{1}{2}b) - \text{coſ}.\,(a + (n+\frac{1}{2})b)}{2\,\text{ſin}.\,\frac{1}{2}b} = \frac{\text{ſin}.\,(a+\frac{1}{2}nb)\,\text{ſin}.\,\frac{1}{2}(n+1)b}{\text{ſin}.\,\frac{1}{2}b}$.

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260. Pari modo, ſi conſideretur ſumma Coſinuum, atque ponatur

$$s = \text{coſ.}\, a + \text{coſ.}\,(a+b) + \text{coſ.}\,(a+2b) + \text{coſ.}\,(a+3b) + \&c.$$

in infinitum, erit $s = \frac{\text{coſ.}\, a + z\,(\text{coſ.}\,(a+b) - 2\,\text{coſ.}\, a.\, \text{coſ.}\, b)}{1 - 2z.\, \text{coſ.}\, b + zz}$, poſito $z = 1$. Quare, ob $2\,\text{coſ.}\, a.\, \text{coſ.}\, b = \text{coſ.}\,(a-b) + \text{coſ.}\,(a+b)$, fiet $s = \frac{\text{coſ.}\, a - \text{coſ.}\,(a-b)}{2\,(1 - \text{coſ.}\, b)}$. At eſt $\text{coſ.}\, f - \text{coſ.}\, g = 2\,\text{ſin.}\,\frac{f+g}{2} \times \text{ſin.}\,\frac{g-f}{2}$; unde erit $\text{coſ.}\, a - \text{coſ.}\,(a-b) = -2\,\text{ſin.}\,(a - \frac{1}{2}b) \times \text{ſin.}\,\frac{1}{2}b$, & ob $1 - \text{coſ.}\, b = 2\,(\text{ſin.}\,\frac{1}{2}b)^2$, erit $s = -\frac{\text{ſin.}\,(a - \frac{1}{2}b)}{2\,\text{ſin.}\,\frac{1}{2}b}$. Quare, cum ſimili modo ſit hujus Seriei

$$\text{coſ.}\,(a+(n+1)b) + \text{coſ.}\,(a+(n+2)b) + \text{coſ.}\,(a+(n+3)b) + \&c.$$

ſumma $= -\frac{\text{ſin.}\,(a+(n+\frac{1}{2})b)}{2\,\text{ſin.}\,\frac{1}{2}b}$, ſi hæc ab illa ſubtrahatur, relinquetur ſumma hujus Seriei

$$s = \text{coſ.}\, a + \text{coſ.}\,(a+b) + \text{coſ.}\,(a+2b) + \text{coſ.}\,(a+3b) + \ldots\ldots + \text{coſ.}\,(a+nb):$$

eritque $s = \frac{-\,\text{ſin.}\,(a - \frac{1}{2}b) + \text{ſin.}\,(a + (n+\frac{1}{2})b)}{2\,\text{ſin.}\,\frac{1}{2}b} = \frac{\text{coſ.}\,(a + \frac{1}{2}nb)\, \text{ſin.}\,\frac{1}{2}(n+1)b}{\text{ſin.}\,\frac{1}{2}b}$.

261. Plurimæ aliæ quæſtiones circa Sinus & Tangentes ex principiis allatis reſolvi poſſent; cujuſmodi ſunt, ſi quadrata, altioreſve Poteſtates Sinuum, Tangentiumve ſummari deberent, verum quia hæc ex reliquis æquationum ſuperiorum coëfficientibus ſimiliter derivantur, iis hic diutius non immoror. Quod autem ad has poſtremas ſummationes attinet, notandum eſt quamcunque Sinuum Coſinuumque Poteſtatem per ſingulos Sinus Coſinuſve explicari poſſe, quod, ut clarius perſpiciatur, breviter exponamus.

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262. Ad hoc expediendum juvabit ex præcedentibus hæc Lemmata depromſiſſe

$$\begin{aligned}
2\,\text{ſin.}\,a.\,\text{ſin.}\,z &= \text{coſ.}\,(a-z) - \text{coſ.}\,(a+z)\\
2\,\text{coſ.}\,a.\,\text{ſin.}\,z &= \text{ſin.}\,(a+z) - \text{ſin.}\,(a-z)\\
2\,\text{ſin.}\,a.\,\text{coſ.}\,z &= \text{ſin.}\,(a+z) + \text{ſin.}\,(a-z)\\
2\,\text{coſ.}\,a.\,\text{coſ.}\,z &= \text{coſ.}\,(a-z) + \text{coſ.}\,(a+z).
\end{aligned}$$

Hinc igitur primum Poteſtates Sinuum reperiuntur

$$\begin{aligned}
\text{ſin.}\,z &= \text{ſin.}\,z\\
2\,(\text{ſin.}\,z)^2 &= 1 - \text{coſ.}\,2z\\
4\,(\text{ſin.}\,z)^3 &= 3\,\text{ſin.}\,z - \text{ſin.}\,3z\\
8\,(\text{ſin.}\,z)^4 &= 3 - 4\,\text{coſ.}\,2z + \text{coſ.}\,4z\\
16\,(\text{ſin.}\,z)^5 &= 10\,\text{ſin.}\,z - 5\,\text{ſin.}\,3z + \text{ſin.}\,5z\\
32\,(\text{ſin.}\,z)^6 &= 10 - 15\,\text{coſ.}\,2z + 6\,\text{coſ.}\,4z - \text{coſ.}\,6z\\
64\,(\text{ſin.}\,z)^7 &= 35\,\text{ſin.}\,z - 21\,\text{ſin.}\,3z + 7\,\text{ſin.}\,5z - \text{ſin.}\,7z\\
128\,(\text{ſin.}\,z)^8 &= 35 - 56\,\text{coſ.}\,2z + 28\,\text{coſ.}\,4z - 8\,\text{coſ.}\,6z + \text{coſ.}\,8z\\
256\,(\text{ſin.}\,z)^9 &= 126\,\text{ſin.}\,z - 84\,\text{ſin.}\,3z + 36\,\text{ſin.}\,5z - 9\,\text{ſin.}\,7z + \text{ſin.}\,9z
\end{aligned}$$

&c.

Lex, qua hi coëfficientes progrediuntur, ex unciis Binomii elevati intelligitur, niſi quod numerus abſolutus in Poteſtatibus paribus ſemiſſis tantum ſit ejus, quem unciæ præbent.

263. Pari modo Poteſtates Coſinuum definientur

$$\begin{aligned}
\text{coſ.}\,z &= \text{coſ.}\,z\\
2\,(\text{coſ.}\,z)^2 &= 1 + \text{coſ.}\,2z\\
4\,(\text{coſ.}\,z)^3 &= 3\,\text{coſ.}\,z + \text{coſ.}\,3z\\
8\,(\text{coſ.}\,z)^4 &= 3 + 4\,\text{coſ.}\,2z + \text{coſ.}\,4z\\
16\,(\text{coſ.}\,z)^5 &= 10\,\text{coſ.}\,z + 5\,\text{coſ.}\,3z + \text{coſ.}\,5z\\
32\,(\text{coſ.}\,z)^6 &= 10 + 15\,\text{coſ.}\,2z + 6\,\text{coſ.}\,4z + \text{coſ.}\,6z\\
64\,(\text{coſ.}\,z)^7 &= 35\,\text{coſ.}\,z + 21\,\text{coſ.}\,3z + 7\,\text{coſ.}\,5z + \text{coſ.}\,7z
\end{aligned}$$

&c.

Hic ratione legis progreſſionis eadem ſunt monenda quæ circa Sinus notavimus.

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CAPUT XV.

De Seriebus ex evolutione Factorum ortis.

264. Sit propoſitum productum ex Factoribus, numero ſive finitis ſive infinitis, conſtans hujuſmodi

$$(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)(1+\epsilon z)(1+\zeta z)\&c.,$$

quod, ſi per multiplicationem actualem evolvatur, det

$$1+Az+Bz^2+Cz^3+Dz^4+Ez^5+Fz^6+\&c.,$$

atque manifeſtum eſt coëfficientes A, B, C, D, E, &c., ita formari ex numeris α, β, γ, δ, ϵ, ζ, &c., ut ſit

$A = \alpha+\beta+\gamma+\delta+\epsilon+\zeta+$ &c. $=$ ſummæ ſingulorum
$B =$ ſummæ Factorum ex binis diverſis
$C =$ ſummæ Factorum ex ternis diverſis
$D =$ ſummæ Factorum ex quaternis diverſis
$E =$ ſummæ Factorum ex quinis diverſis
&c.

donec perveniatur ad productum ex omnibus.

265. Quod ſi ergo ponatur $z=1$, productum hoc

$$(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\epsilon)\&c.$$

æquabitur unitati cum Serie numerorum omnium, qui ex his α, β, γ, δ, ϵ, &c., vel ſumendis ſingulis, vel duobus pluribuſve diverſis in ſe multiplicandis, naſcuntur. Atque ſi idem numerus duobus pluribuſve modis reſultare queat, etiam idem bis pluriesve in hac numerorum Serie occurret.

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LIB. I. 266. Si ponatur $z = -1$, productum hoc

$$(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\epsilon)\ \&c.$$

æquabitur unitati cum Serie numerorum omnium, qui ex his α, β, γ, δ, ϵ, ζ, &c. vel ſumendis ſingulis, vel duobus pluribusve diverſis in ſe multiplicandis, naſcuntur; ut ante quidem, verum hoc diſcrimine, ut ii numeri, qui vel ex ſingulis, vel ternis, vel quinis, vel numero imparibus naſcuntur, ſint negativi, illi vero, qui vel ex binis, vel quaternis, vel ſenis vel numero paribus reſultant, ſint affirmativi.

267. Scribantur pro α, β, γ, δ, &c., numeri primi omnes 2, 3, 5, 7, 11, 13, &c., atque hoc productum

$$(1+2)(1+3)(1+5)(1+7)(1+11)(1+13)\ \&c. = P$$

æquabitur unitati, cum Serie omnium numerorum vel primorum ipſorum, vel ex primis diverſis per multiplicationem ortorum. Erit ergo

$$P = 1+2+3+5+6+7+10+11+13+14+15+17+\&c.,$$

in qua Serie omnes occurrunt numeri naturales, exceptis Poteſtatibus, iiſque qui per quamvis Poteſtatem ſunt diviſibiles. Deſunt ſcilicet numeri 4, 8, 9, 12, 16, 18, &c., quoniam ſunt vel Poteſtates, ut 4, 8, 9, 16, &c., vel per Poteſtates diviſibiles ut 12, 18, &c.

268. Simili modo res ſe habebit, ſi pro α, β, γ, δ, &c.. Poteſtates quæcunque numerorum primorum ſubſtituantur. Scilicet ſi ponamus

$$P = \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\&c..$$

Erit enim multiplicatione inſtituta:

$$P = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{10^n}+\frac{1}{11^n}+\&c.$$

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in quibus fractionibus omnes occurrunt numeri præter illos qui vel ipſi ſunt Poteſtates, vel per Poteſtatem quampiam diviſibiles. Cum enim omnes numeri integri ſint vel primi vel ex primis per multiplicationem compoſiti, hic ii tantum numeri excludentur, in quorum formationem idem numerus primus bis vel pluries ingreditur. CAP. XV.

269. Si numeri α, β, γ, δ, &c., negative capiantur, ut ante (266) fecimus, atque ponatur

$$P = \left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right) \text{ \&c., erit}$$

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \frac{1}{13^n} + \frac{1}{15^n} - \text{\&c.},$$

ubi iterum, ut ante, omnes occurrunt numeri præter Poteſtates ac diviſibiles per Poteſtates. Verum ipſi numeri primi, & qui ex ternis, quinis, numerove imparibus conſtant, ſignum habent præfixum —, qui autem ex binis, vel quaternis, vel ſenis, vel numero paribus formantur, ſignum habent +. Sic in hac Serie occurret terminus $\frac{1}{30^n}$, quia eſt $30 = 2 \cdot 3 \cdot 5$, neque adeo Poteſtatem complectitur, habebit vero hic terminus $\frac{1}{30^n}$ ſignum —, quia 30 eſt productum ex tribus numeris primis.

270. Conſideremus jam hanc expreſſionem

$$\frac{1}{(1 - \alpha z)(1 - \beta z)(1 - \gamma z)(1 - \delta z)(1 - \varepsilon z) \text{ \&c.}}$$

quæ per diviſionem actualem evoluta præbeat hanc Seriem:

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$$1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \&c.$$

atque manifeſtum eſt coëfficientes A, B, C, D, E, &c., ſequenti modo ex numeris, $\alpha, \beta, \gamma, \delta, \epsilon$, &c., componi, ut ſit

A = ſummæ ſingulorum
B = ſummæ Factorum ex binis
C = ſummæ Factorum ex ternis
D = ſummæ Factorum ex quaternis
&c.

} non excluſis Factoribus iiſdem.

271. Poſito ergo $z = 1$, iſta expreſſio

$$\frac{1}{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\epsilon)\ \&c.}$$

æquabitur unitati cum Serie numerorum omnium, qui ex his $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, &c., vel ſumendis ſingulis, vel duobus pluribuſve in ſe multiplicandis, oriuntur, non excluſis æqualibus. Hoc ergo differt iſta numerorum Series ab illa, quæ (§. 265) prodiit, quod ibi Factores tantum diverſi ſumi debebant, hic autem idem Factor bis plurieſve occurrere poſſit. Hic ſcilicet omnes numeri occurrunt, qui per multiplicationem ex his $\alpha, \beta, \gamma, \delta$, &c., provenire poſſunt.

272. Hanc ob rem Series ſemper ex terminorum numero infinito conſtat, ſive Factorum numerus fuerit infinitus, ſive finitus. Sic erit

$$\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \&c.,$$

ubi omnes numeri adſunt, qui ex binario ſolo per multiplicationem oriuntur, ſeu omnes binarii Poteſtates. Deinde erit

$$\frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \&c.,$$

ubi

ubi alii numeri non occurrunt, niſi qui ex his duobus 2 & 3 per multiplicationem originem trahunt; ſeu qui alios Diviſores præter 2 & 3 non habent.

273. Si igitur pro α, β, γ, δ, &c., unitas per ſingulos omnes numeros primos diviſa ſcribatur, ac ponatur

$$P = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1-\frac{1}{13})\&c.},$$

fiet

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi omnes numeri tam primi, quam qui ex primis per multiplicationem naſcuntur, occurrunt. Cum autem omnes numeri vel ſint ipſi primi, vel ex primis per multiplicationem oriundi, manifeſtum eſt, hic omnes omnino numeros integros in denominatoribus adeſſe debere.

274. Idem evenit, ſi numerorum primorum Poteſtates quæcunque accipiantur: ſi enim ponatur

$$P = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\&c.},$$

fiet

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

ubi omnes numeri naturales nullo excepto occurrunt. Quod ſi autem in Factoribus ubique ſignum + ſtatuatur, ut ſit

$$P = \frac{1}{(1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\&c.},$$

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c.,$$

ubi numeri primi habent ſignum —; qui ſunt producti ex duobus primis, ſive iiſdem ſive diverſis, ſignum habent +; & generatim, quorum numerorum numerus Factorum primorum eſt par, ſignum habent +, qui autem ex Factoribus primis numero imparibus conſtant, habent ſignum —. Sic terminus $\frac{1}{240^n}$, ob $240 = 2.2.2.2.3.5$, habebit ſignum +, cujus legis ratio percipitur ex §. 270, ſi ponatur $z = -1$.

275. Si hæc cum ſuperioribus conferantur, naſcentur binæ Series quarum productum unitati æquatur. Sit enim

$$P = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\&c.},$$

&

$$Q = (1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\&c.,$$

erit

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

$$Q = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n}\ \&c.$$

(269), atque manifeſtum eſt fore $PQ = 1$.

276. Sin autem ponatur

$$P = \frac{1}{(1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\ \&c.},$$

&

$$Q = (1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\ \&c.,$$

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \&c.$$

$$Q = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$$

ſimilique modo habebitur $PQ = 1$. Cognita ergo alterius Seriei ſumma, ſimul alterius innoteſcet.

277. Viciſſim porro ex cognitis ſummis harum Serierum; aſſignari poterunt valores Factorum infinitorum. Sit nimirum

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \&c.$$

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \frac{1}{7^{2n}} + \&c.,$$

eritque

$$M = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\ \&c.}$$

$$N = \frac{1}{(1-\frac{1}{2^{2n}})(1-\frac{1}{3^{2n}})(1-\frac{1}{5^{2n}})(1-\frac{1}{7^{2n}})(1-\frac{1}{11^{2n}})\ \&c.}$$

Hinc per diviſionem naſcitur

$$\frac{M}{N} = (1+\frac{1}{2^n})(1+\frac{1}{3^n})(1+\frac{1}{5^n})(1+\frac{1}{7^n})(1+\frac{1}{11^n})\ \&c.$$

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denique vero erit

$$\frac{MM}{N} = \frac{2^n+1}{2^n-1} \cdot \frac{3^n+1}{3^n-1} \cdot \frac{5^n+1}{5^n-1} \cdot \frac{7^n+1}{7^n-1} \cdot \frac{11^n+1}{11^n-1} \cdot \&c.$$

Ex cognitis ergo M & N, præter valores horum productorum, ſummæ harum Serierum habebuntur

$$\frac{1}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \&c.$$

$$\frac{1}{N} = 1 - \frac{1}{2^{2n}} - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{6^{2n}} - \frac{1}{7^{2n}} + \frac{1}{10^{2n}} - \frac{1}{11^{2n}} - \&c.$$

$$\frac{M}{N} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$$

$$\frac{N}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c.,$$

ex quarum combinatione multæ aliæ deduci poſſunt.

EXEMPLUM I.

Sit $n = 1$, &, quoniam ſupra demonſtravimus eſſe, $l\frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \&c.$, erit, poſito $x=1$, $l\frac{1}{1-1} = l\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.$ At Logarithmus numeri infinite magni ∞ ipſe eſt infinite magnus, ex quo erit

$M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \&c. = \infty.$

Hinc ob $\frac{1}{M} = \frac{1}{\infty} = 0$, fiet

$$0 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \frac{1}{14} + \frac{1}{15} \ \&c.$$

Tum verò in productis habebitur

$$M = \infty = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})\&c.},$$

unde fit

$$\infty = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \&c.,$$

&

$$0 = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} \cdot \frac{16}{17} \cdot \frac{18}{19} \cdot \&c.$$

Deinde per ſummationem Serierum ſupra traditam erit

$N = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \&c. = \frac{\pi\pi}{6}$, hinc obtinentur iſtæ ſummæ Serierum

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \&c.$$

$$\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \&c.$$

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} \ \&c.$$

Denique pro Factoribus orietur

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LIB. I. $\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1}$. &c.,

ſeu

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{25}{24} \cdot \frac{49}{48} \cdot \frac{121}{120} \cdot \frac{169}{168} \cdot \text{\&c.}$$

&, ob $\frac{M}{N} = \infty$ ſeu $\frac{N}{M} = 0$, habebitur

$$\infty = \frac{3}{2} \cdot \frac{4}{4} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} \cdot \frac{20}{19} \cdot \text{\&c.},$$

ſeu

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \text{\&c.},$$

atque

$$\infty = \frac{3}{1} \cdot \frac{4}{2} \cdot \frac{6}{4} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{14}{12} \cdot \frac{18}{16} \cdot \frac{20}{18} \cdot \text{\&c.},$$

ſeu

$$0 = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{9}{10} \cdot \text{\&c.},$$

quarum fractionum (excepta prima) numeratores unitate deficiunt a denominatoribus, ſummæ autem ex numeratoribus & denominatoribus cujuſque fractionis conſtanter præbent numeros primos, 3, 5, 7, 11, 13, 17, 19, &c.

EXEMPLUM II.

Sit $n = 2$, eritque ex ſuperioribus

$$M = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \text{\&c.} = \frac{\pi\pi}{6}$$

$$N = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \text{\&c.} = \frac{\pi^4}{90}$$

Hinc primo iſtæ Series ſummantur

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$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \text{\&c.}$$

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} - \text{\&c.}$$

$$\frac{15}{\pi^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{11^2} + \text{\&c.}$$

$$\frac{\pi\pi}{15} = 1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2}\ \text{\&c.},$$

Deinde valores ſequentium productorum innoteſcunt

$$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot \text{\&c.}$$

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \text{\&c.}$$

$$\frac{15}{\pi\pi} = \frac{2^2+1}{2^2} \cdot \frac{3^2+1}{3^2} \cdot \frac{5^2+1}{5^2} \cdot \frac{7^2+1}{7^2} \cdot \frac{11^2+1}{11^2} \cdot \text{\&c.},$$

ſeu

$$\frac{\pi\pi}{15} = \frac{4}{5} \cdot \frac{9}{10} \cdot \frac{25}{26} \cdot \frac{49}{50} \cdot \frac{121}{122} \cdot \frac{169}{170} \cdot \text{\&c.},$$

&

$$\frac{5}{2} = \frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \frac{11^2+1}{11^2-1} \cdot \text{\&c.},$$

ſive

$$\frac{5}{2} = \frac{5}{3} \cdot \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \text{\&c.},$$

vel

$$\frac{3}{2} = \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \text{\&c.}.$$

In his fractionibus numeratores unitate ſuperant denominatores, ſimul vero ſumti præbent quadrata numerorum primorum 3^2, 5^2, 7^2, 11^2, &c.

EXEMPLUM III.

Quia ex ſuperioribus valores ipſius M tantum ſi n ſit numerus par, aſſignare licet, ponamus $n = 4$, eritque

$$M = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \&c. = \frac{\pi^4}{90}$$

$$N = 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} + \&c. = \frac{\pi^8}{9450}$$

Hinc primæ ſequentes Series ſummantur

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4}\ \&c.$$

$$\frac{9450}{\pi^8} = 1 - \frac{1}{2^8} - \frac{1}{3^8} - \frac{1}{5^8} + \frac{1}{6^8} - \frac{1}{7^8} + \frac{1}{10^8} - \frac{1}{11^8}\ \&c.$$

$$\frac{105}{\pi^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{10^4} + \frac{1}{11^4}\ \&c.$$

$$\frac{\pi^4}{105} = 1 - \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} - \frac{1}{8^4} + \frac{1}{9^4}\ \&c.,$$

Deinde etiam valores ſequentium productorum obtinentur

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \&c.$$

$$\frac{\pi^8}{9450} = \frac{2^8}{2^8-1} \cdot \frac{3^8}{3^8-1} \cdot \frac{5^8}{5^8-1} \cdot \frac{7^8}{7^8-1} \cdot \frac{11^8}{11^8-1} \cdot \&c.$$

$$\frac{105}{\pi^4} = \frac{2^4+1}{2^4} \cdot \frac{3^4+1}{3^4} \cdot \frac{5^4+1}{5^4} \cdot \frac{7^4+1}{7^4} \cdot \frac{11^4+1}{11^4} \cdot \&c.$$

&

$$\frac{7}{6} = \frac{2^4+1}{2^4-1} \cdot \frac{3^4+1}{3^4-1} \cdot \frac{5^4+1}{5^4-1} \cdot \frac{7^4+1}{7^4-1} \cdot \frac{11^4+1}{11^4-1} \cdot \&c.$$

ſeu

$$\frac{35}{34} = \frac{41}{40} \cdot \frac{313}{312} \cdot \frac{1201}{1200} \cdot \frac{7321}{7320} \cdot \&c.,$$

in

in his Factoribus numeratores unitate ſuperant denominatores, ſimul vero ſumti præbent bi-quadrata numerorum primorum imparium 3, 5, 7, 11, &c.

278. Quoniam hic ſummam Seriei

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \&c.$$

ad Factores reduximus, ad Logarithmos commode progredi licebit. Nam, cum ſit

$$M = \frac{1}{(1-\frac{1}{2^n})(1-\frac{1}{3^n})(1-\frac{1}{5^n})(1-\frac{1}{7^n})(1-\frac{1}{11^n})\&c.},$$

erit

$$lM = -l(1-\frac{1}{2^n}) - l(1-\frac{3}{3^n}) - l(1-\frac{1}{5^n}) - l(1-\frac{1}{7^n}) - \&c..$$

Hinc, ſumendis Logarithmis hyperbolicis, erit

$$\begin{aligned} lM = &+ 1\left(\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \&c.\right) \\ &+ \frac{1}{2}\left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \&c.\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \&c.\right) \\ &+ \frac{1}{4}\left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \&c.\right) \\ &\&c. \end{aligned}$$

Quod ſi inſuper ponamus

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \&c.,$$

ut ſit

$$N = \frac{1}{(1-\frac{1}{2^{2n}})(1-\frac{1}{3^{2n}})(1-\frac{1}{5^{2n}})(1-\frac{1}{7^{2n}})(1-\frac{1}{11^{2n}})\,\&c.},$$

fiet, Logarithmis hyperbolicis ſumendis,

$$\begin{aligned} lN = &+1\left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \&c.\right) \\ &+\frac{1}{2}\left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \&c.\right) \\ &+\frac{1}{3}\left(\frac{1}{2^{6n}} + \frac{1}{3^{6n}} + \frac{1}{5^{6n}} + \frac{1}{7^{6n}} + \frac{1}{11^{6n}} + \&c.\right) \\ &+\frac{1}{4}\left(\frac{1}{2^{8n}} + \frac{1}{3^{8n}} + \frac{1}{5^{8n}} + \frac{1}{7^{8n}} + \frac{1}{11^{8n}} + \&c.\right) \\ &\quad\&c. \end{aligned}$$

Ex his conjunctis fiet $lM - \frac{1}{2}\,lN =$

$$\begin{aligned} &+1\left(\frac{1}{2^{n}} + \frac{1}{3^{n}} + \frac{1}{5^{n}} + \frac{1}{7^{n}} + \frac{1}{11^{n}} + \&c.\right) \\ &+\frac{1}{3}\left(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \&c.\right) \\ &+\frac{1}{5}\left(\frac{1}{2^{5n}} + \frac{1}{3^{5n}} + \frac{1}{5^{5n}} + \frac{1}{7^{5n}} + \frac{1}{11^{5n}} + \&c.\right) \\ &+\frac{1}{7}\left(\frac{1}{2^{7n}} + \frac{1}{3^{7n}} + \frac{1}{5^{7n}} + \frac{1}{7^{7n}} + \frac{1}{11^{7n}} + \&c.\right) \\ &\quad\&c. \end{aligned}$$

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279. Si $n = 1$, erit $M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c. = l\infty$, & $N = \frac{\pi\pi}{6}$; hincque erit $l\infty - \frac{1}{2}\, l\frac{\pi\pi}{6} =$

$$+ 1 \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c.\right)$$
$$+ \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} + \&c.\right)$$
$$+ \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{11^5} + \&c.\right)$$
$$+ \frac{1}{7}\left(\frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{11^7} + \&c.\right)$$
$$\&c.$$

Verum hæ Series, præter primam, non ſolum ſummas habent finitas, ſed etiam cunctæ ſimul ſumtæ ſummam efficiunt finitam, eamque ſatis parvam: unde neceſſe eſt ut Seriei primæ $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c.$, ſumma ſit infinite magna, quantitate ſcilicet ſatis parva deficiet a Logarithmo hyperbolico Seriei $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$

280. Sit $n = 2$; erit $M = \frac{\pi\pi}{6}$ & $N = \frac{\pi^4}{90}$: unde fit

$$2\,l\pi - l6 = 1\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c.\right)$$
$$+ \frac{1}{2}\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \&c.\right)$$
$$+ \frac{1}{3}\left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \&c.\right)$$
$$\&c.$$

$$4l\pi - l90 = +\ 1\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \&c.\right)$$
$$+ \frac{1}{2}\left(\frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{11^8} + \&c.\right)$$
$$+ \frac{1}{3}\left(\frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \frac{1}{11^{12}} + \&c.\right)$$
&c.

$$lM - \frac{1}{2}lN = \frac{1}{2}l\frac{5}{2} = 1\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c.\right)$$
$$+ \frac{1}{3}\left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \&c.\right)$$
$$+ \frac{1}{5}\left(\frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \frac{1}{11^{10}} + \&c.\right)$$
&c.

281. Quanquam lex, qua numeri primi progrediuntur, non conſtat, tamen harum Serierum altiorum Poteſtatum ſummæ non difficulter proxime aſſignari poterunt. Sit enim hæc Series

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \&c.,$$
&
$$S = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \&c.,$$
erit
$$S = M - 1 - \frac{1}{4^n} - \frac{1}{6^n} - \frac{1}{8^n} - \frac{1}{9^n} - \frac{1}{10^n} - \&c.,$$
& ob
$$\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \frac{1}{12^n} + \&c.,$$
erit
$$S = M - \frac{M}{2^n} - 1 + \frac{1}{2^n} - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \&c.;$$
ſeu

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$$S = (M - 1)\left(1 - \frac{1}{2^n}\right) - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \frac{1}{25^n} - \frac{1}{27^n} - \&c.,$$

& ob

$$M\left(1 - \frac{1}{2^n}\right)\frac{1}{3^n} = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \&c.,$$

erit

$$S = (M - 1)\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right) + \frac{1}{6^n} - \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{45^n} - \&c..$$

Hinc, ob datam ſummam M, valor ipſius S commode invenitur, ſi quidem n fuerit numerus mediocriter magnus.

282. Inventis autem ſummis altiorum Poteſtatum, etiam ſummæ Poteſtatum minorum ex formulis inventis exhiberi poſſunt. Atque hac methodo ſequentes prodierunt ſummæ Seriei

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

ſi ſit	erit ſumma Seriei
$n = 2$;	0, 452247420041222
$n = 4$;	0, 076993139764252
$n = 6$;	0, 017070086850639
$n = 8$;	0, 004061405366515
$n = 10$;	0, 000993603573633
$n = 12$;	0, 000246026470033
$n = 14$;	0, 000061244396725
$n = 16$;	0, 000015282026219
$n = 18$;	0, 000003817278702

$n = 20$;	0, 000000953961123
$n = 22$;	0, 000000238450446
$n = 24$;	0, 000000059608184
$n = 26$;	0, 000000014901555
$n = 28$;	0, 000000003725333
$n = 30$;	0, 000000000931323
$n = 32$;	0, 000000000232830
$n = 34$;	0, 000000000058207
$n = 36$;	0, 000000000014551

reliquæ ſummæ parium Poteſtatum in ratione quadrupla decreſcunt.

283. Hæc autem Seriei $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} +$ &c., in productum infinitum converſio etiam directe inſtitui poteſt hoc modo: ſit

$$A = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \text{\&c., ſubtrahe}$$

$$\frac{1}{2^n} A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \text{\&c.,}$$

erit

$$\left(1 - \frac{1}{2^n}\right) A = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{\&c.}$$

$= B$: ſic ſublati ſunt omnes termini per 2 diviſibiles,

ſubtr. $\frac{1}{3^n} B = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} +$ &c.,

erit

$$\left(1 - \frac{1}{3^n}\right) B = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \text{\&c.} = C:$$

ſic inſuper ſublati ſunt omnes termini per 3 diviſibiles,

ſubtr. $\frac{1}{5^n} C = \quad \frac{1}{5^n} + \frac{1}{25^n} + \frac{1}{35^n} + \frac{1}{55^n} + \&c.,$

erit

$$(1 - \frac{1}{5^n}) C = 1 + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

ſic ſublati etiam ſunt omnes termini per 5 diviſibiles. Pari modo tolluntur termini diviſibiles per 7, 11, reliquoſque numeros primos; manifeſtum autem eſt ſublatis omnibus terminis, qui per numeros primos diviſibiles ſint, ſolam unitatem relinqui. Quare pro B, C, D, E, &c., valoribus reſtitutis tandem orietur

$$A(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n})\&c. = 1,$$

unde Seriei propoſitæ ſumma erit =

$$A = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n})\&c.},$$

ſeu

$$A = \frac{2^n}{2^n - 1} \cdot \frac{3^n}{3^n - 1} \cdot \frac{5^n}{5^n - 1} \cdot \frac{7^n}{7^n - 1} \cdot \frac{11^n}{11^n - 1} \cdot \&c.$$

284. Hæc methodus jam commode adhiberi poterit ad alias Series, quarum ſummas ſupra invenimus, in producta infinita convertendas. Invenimus autem ſupra (175) ſummas harum Serierum

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \&c.,$$

ſi n fuerit numerus impar, ſumma enim eſt $= N\pi^n$ & valores ipſius N loco citato dedimus. Notandum autem eſt

LIB. I. cum hic tantum numeri impares occurrunt, eos qui ſint formæ $4m+1$ habere ſignum $+$, reliquos formæ $4m-1$ ſignum $-$. Sit igitur

$$A = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{15^n} + \&c.$$

$$\frac{1}{3^n} A = \frac{1}{3^n} - \frac{1}{9^n} + \frac{1}{15^n} - \frac{1}{21^n} + \frac{1}{27^n} - \&c.,$$ addatur, erit

$$\left(1 + \frac{1}{3^n}\right) A = 1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} \&c., = B$$

$$\frac{1}{5^n} B = \frac{1}{5^n} + \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{55^n} \&c.,$$ ſubtrahatur, erit

$$\left(1 - \frac{1}{5^n}\right) B = 1 - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \&c. = C,$$

ubi jam numeri per 3 & 5 diviſibiles deſunt,

$$\frac{1}{7^n} C = \frac{1}{7^n} - \frac{1}{49^n} - \frac{1}{77^n} + \&c.,$$ addatur, erit

$$\left(1 + \frac{1}{7^n}\right) C = 1 - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \&c. = D,$$

ſic etiam numeri per 7 diviſibiles ſunt ſublati,

$$\frac{1}{11^n} D = \frac{1}{11^n} - \frac{1}{121^n} + \&c.,$$ addatur, erit

$$\left(1 + \frac{1}{11^n}\right) D = 1 + \frac{1}{13^n} + \frac{1}{17^n} - \&c. = E,$$

ſic numeri per 11 diviſibiles quoque ſunt ſublati. Auferendis autem

autem hoc modo reliquis numeris omnibus per reliquos numeros primos diviſibilibus, tandem prodibit

$$A\left(1+\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\left(1-\frac{1}{13^n}\right) \&c. = 1,$$

ſeu

$$A = \frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n-1} \cdot \frac{7^n}{7^n+1} \cdot \frac{11^n}{11^n+1} \cdot \frac{13^n}{13^n-1} \times \frac{17^n}{17^n-1} \cdot \&c.,$$

ubi in numeratoribus occurrunt Poteſtates omnium numerorum primorum, quæ in denominatoribus inſunt unitate ſive auctæ ſive minutæ, prout numeri primi fuerint formæ $4m-1$, vel $4m+1$.

285. Poſito ergo $n=1$, ob $A=\frac{\pi}{4}$, erit

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdot \&c.,$$

ſupra autem invenimus eſſe

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{3^2}{2.4} \cdot \frac{5^2}{4.6} \cdot \frac{7^2}{6.8} \cdot \frac{11^2}{10.12} \cdot \frac{13^2}{12.14} \cdot \frac{17^2}{16.18} \cdot \frac{19^2}{18.20} \cdot \&c..$$

Dividatur ſecunda per primam & orietur

$$\frac{2\pi}{3} = \frac{4}{3} \cdot \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \&c,$$

ſeu

$$\frac{\pi}{2} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \&c.,$$

ubi numeri primi conſtituunt numeratores, denominatores vero ſunt numeri impariter pares, unitate differentes a nume-

 ratoribus. Quod ſi hæc denuo per primam $\frac{\pi}{4}$ dividatur, erit

$$2 = \frac{4}{2}\cdot\frac{4}{6}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\frac{16}{18}\cdot\frac{20}{18}\cdot\frac{24}{22}\cdot \&c.,$$

ſeu

$$2 = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{12}{11}\cdot \&c.,$$

quæ fractiones oriuntur ex numeris primis imparibus 3, 5, 7, 11, 13, 17, &c., quemque in duas partes unitate differentes diſpeſcendo, & partes pares pro numeratoribus, impares pro denominatoribus ſumendo.

286. Si hæ expreſſiones cum *Walliſiana* comparentur

$$\frac{\pi}{2} = \frac{2.\,2.\,4.\,4.\,6.\,6.\,8.\,8.\,10.\,10.\,12}{1.\,3.\,3.\,5.\,5.\,7.\,7.\,9.\,9.\,11.\,11}\ \&c.,$$

ſeu

$$\frac{4}{\pi} = \frac{3.\,3}{2.\,4}\cdot\frac{5.\,5}{4.\,6}\cdot\frac{7.\,7}{6.\,8}\cdot\frac{9.\,9}{8.\,10}\cdot\frac{11.\,11}{10.\,12}\cdot \&c.,$$

cum ſit

$$\frac{\pi\pi}{8} = \frac{3.\,3}{2.\,4}\cdot\frac{5.\,5}{4.\,6}\cdot\frac{7.\,7}{6.\,8}\cdot\frac{11.\,11}{10.\,12}\cdot\frac{13.\,13}{12.\,14}\cdot \&c.,$$

illa per hanc diviſa dabit

$$\frac{32}{\pi^3} = \frac{9.\,9}{8.\,10}\cdot\frac{15.\,15}{14.\,16}\cdot\frac{21.\,21}{20.\,22}\cdot\frac{25.\,25}{24.\,26}\cdot \&c.,$$

ubi in numeratoribus occurrunt omnes numeri impares non primi.

287. Sit jam $n = 3$, erit $A = \frac{\pi^3}{32}$, unde fit

$$\frac{\pi^3}{32} = \frac{3^3}{3^3+1}\cdot\frac{5^3}{5^3-1}\cdot\frac{7^3}{7^3+1}\cdot\frac{11^3}{11^3+1}\cdot\frac{13^3}{13^3-1}\cdot\frac{17^3}{17^3-1}\cdot$$

&c.. At ex Serie

$$\frac{\pi^6}{945} = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \&c.,$$

fit

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$$\frac{\pi^6}{945} = \frac{2^6}{2^6-1} \cdot \frac{3^6}{3^6-1} \cdot \frac{5^6}{5^6-1} \cdot \frac{7^6}{7^6-1} \cdot \frac{11^6}{11^6-1} \cdot \frac{13^6}{13^6-1} \cdot \text{\&c., ſeu}$$

$$\frac{\pi^6}{960} = \frac{3^6}{3^6-1} \cdot \frac{5^6}{5^6-1} \cdot \frac{7^6}{7^6-1} \cdot \frac{11^6}{11^6-1} \cdot \frac{13^6}{13^6-1} \cdot \text{\&c.},$$

quæ per primam diviſa dabit

$$\frac{\pi^3}{30} = \frac{3^3}{3^3-1} \cdot \frac{5^3}{5^3+1} \cdot \frac{7^3}{7^3-1} \cdot \frac{11^3}{11^3-1} \cdot \frac{13^3}{13^3+1} \cdot \frac{17^3}{17^3+1} \cdot \text{\&c.},$$

hæc vero denuo per primam diviſa dabit

$$\frac{16}{15} = \frac{3^3+1}{3^3-1} \cdot \frac{5^3-1}{5^3+1} \cdot \frac{7^3+1}{7^3-1} \cdot \frac{11^3+1}{11^3-1} \cdot \frac{13^3-1}{13^3+1} \cdot \frac{17^3-1}{17^3+1} \cdot \text{\&c., ſeu}$$

$$\frac{16}{15} = \frac{14}{13} \cdot \frac{62}{63} \cdot \frac{172}{171} \cdot \frac{666}{665} \cdot \frac{1098}{1099} \cdot \text{\&c.},$$

quæ fractiones formantur ex cubis numerorum primorum imparium, quemque in duas partes unitate differentes diſpeſcendo, ac partes pares pro numeratoribus, impares pro denominatoribus ſumendo.

288. Ex his expreſſionibus denuo novæ Series formari poſſunt, in quibus omnes numeri naturales denominatores conſtituunt. Cnm enim ſit

$$\frac{\pi}{4} = \frac{3}{3+1} \cdot \frac{5}{5-1} \cdot \frac{7}{7+1} \cdot \frac{11}{11+1} \cdot \frac{13}{13-1} \cdot \text{\&c.},$$

erit

$$\frac{\pi}{6} = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13})\text{\&c.}},$$

unde per evolutionem hæc Series naſcetur

$$\frac{\pi}{6} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \text{\&c.},$$

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ubi ratio ſignorum ita eſt comparata, ut binarius habeat —, numeri primi formæ $4m-1$ ſignum — & numeri primi formæ $4m+1$ ſignum +; numeri autem compoſiti ea habent ſigna, quæ ipſis ratione multiplicationis ex primis conveniunt. Sic. patebit ſignum fractionis $\frac{1}{60}$, ob 60 =

$$\overset{-}{2}\cdot\overset{-}{2}\cdot\overset{-}{3}\cdot\overset{+}{5},$$

quod erit —. Simili modo porro erit

$$\frac{\pi}{2}=\frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13})\,\&c.},$$

unde orietur hæc Series

$$\frac{\pi}{2}=1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}\,\&c.,$$

ubi binarius habet ſignum +, numeri primi formæ $4m-1$ ſignum —, numeri primi formæ $4m+1$ ſignum + & numerus quiſque compoſitus id habet ſignum, quod ipſi ratione compoſitionis ex primis convenit, ſecundum regulas multiplicationis.

289. Cum deinde ſit

$$\frac{\pi}{2}=\frac{1}{(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

erit per evolutionem

$$\frac{\pi}{2}=1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}\,\&c.,$$

ubi tantum numeri impares occurrunt, ſigna autem ita ſunt comparata, ut numeri primi formæ $4m-1$ ſignum habeant +, numeri primi formæ $4m+1$ ſignum —, unde ſimul numerorum compoſitorum ſigna definiuntur. Binæ porro

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Series hinc formari poſſunt, ubi omnes numeri occurrunt; erit ſcilicet

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

unde per evolutionem oritur

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\ \&c.,$$

ubi binarius ſignum habet $+$, numeri primi formæ $4m-1$ ſignum $+$, numeri vero primi formæ $4m+1$ ſignum $-$. Tum vero etiam erit

$$\frac{\pi}{3} = \frac{1}{(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

unde per evolutionem oritur

$$\frac{\pi}{3} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\ \&c.,$$

ubi binarius habet ſignum $-$, numeri primi formæ $4m-1$ ſignum $+$ & numeri primi formæ $4m+1$ ſignum $-$.

290. Poſſunt hinc etiam innumerabiles aliæ ſignorum conditiones exhiberi, ita ut Seriei

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \&c.,$$

ſumma aſſignari queat. Cum ſcilicet ſit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})\,\&c.}.$$

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LIB. I.

Multiplicetur hæc expreſſio per $\frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$, erit

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})\ \&c.},$$

&

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11}\ \&c.,$$

ubi binarius ſignum habet $+$; ternarius $+$; reliqui numeri primi omnes formæ $4m - 1$ ſignum $-$; at numeri primi formæ $4m + 1$ ſignum $+$; unde pro numeris compoſitis ratio ſignorum intelligitur. Simili modo, cum ſit

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})\ \&c.},$$

multiplicetur per $\frac{1+\frac{1}{5}}{1-\frac{1}{5}} = \frac{3}{2}$, erit

$$\frac{3\pi}{2} = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})(1+\frac{1}{17})\ \&c.},$$

unde per evolutionem oritur

$$\frac{3\pi}{2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi binarius habet ſignum $+$, numeri primi formæ $4m - 1$ ſignum $+$ & numeri primi formæ $4m + 1$, præter quinarium, ſignum $-$.

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291. Poſſunt etiam innumerabiles hujuſmodi Series exhiberi, quarum ſumma ſit $= 0$. Cum enim ſit

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \&c.,$$

erit

$$0 = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

unde, ut ſupra vidimus, oritur

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\ \&c.,$$

ubi omnes numeri primi ſignum habent $-$; compoſitorumque numerorum ſigna regulam multiplicationis ſequuntur. Multiplicemus autem illam expreſſionem per $\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 3$, erit pariter

$$0 = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\,\&c.},$$

unde per evolutionem naſcitur

$$0 = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\ \&c.,$$

ubi binarius habet ſignum $+$; reliqui numeri primi omnes ſignum $-$. Simili modo quoque erit

LIB. I. $$0 = \frac{1}{(1+\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13})\&c.},$$

unde oritur iſta Series

$$0 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\ \&c.,$$

ubi omnes numeri primi, præter 3 & 5, habent ſignum —. In genere autem notandum eſt, quoties omnes numeri primi, exceptis tantum aliquibus, habebunt ſignum —, ſummam Seriei fore $= 0$. Contra autem quoties omnes numeri primi, exceptis tantum aliquibus, habebunt ſignum +, tum ſummam Seriei fore infinite magnam.

292. Supra etiam (176) ſummam dedimus Seriei

$$A = 1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{10^n} - \frac{1}{11^n} + \frac{1}{13^n}\ \&c.,$$

ſi fuerit n numerus impar: Erit ergo

$$\frac{1}{2^n}A = \frac{1}{2^n} - \frac{1}{4^n} + \frac{1}{8^n} - \frac{1}{10^n} + \frac{1}{14^n} - \&c.,$$

quæ addita dat

$$B = (1 + \frac{1}{2^n})A = 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} + \frac{1}{25^n} - \&c.$$

$$\frac{1}{5^n}B = \frac{1}{5^n} - \frac{1}{25^n} + \frac{1}{35^n} - \frac{1}{55^n}\ \&c.,$$ addatur,

erit

$C =$

$$C = \left(1 + \frac{1}{5^n}\right) B = 1 + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} \ \&c.$$ CAP. XV.

$$\frac{1}{7^n} C = \frac{1}{7^n} + \frac{1}{49^n} - \frac{1}{77^n} + \&c.,$$ ſubtrahatur, erit

$$D = \left(1 - \frac{1}{7^n}\right) C = 1 - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \&c..$$

Ex his tandem fiet

$$A\left(1 + \frac{1}{2^n}\right)\left(1 + \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 + \frac{1}{11^n}\right)\left(1 - \frac{1}{13^n}\right) \&c. = 1,$$

ubi numeri primi unitate excedentes multipla ſenarii habent ſignum —, deficientes autem ſignum +. Eritque

$$A = \frac{2^n}{2^n+1} \cdot \frac{5^n}{5^n+1} \cdot \frac{7^n}{7^n-1} \cdot \frac{11^n}{11^n+1} \cdot \frac{13^n}{13^n-1} \cdot \&c..$$

293. Conſideremus caſum $n = 1$, quo $A = \frac{\pi}{3\sqrt{3}}$; eritque

$$\frac{\pi}{3\sqrt{3}} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \&c.,$$

ubi in numeratoribus poſt 3 occurrunt omnes numeri primi, denominatores vero a numeratoribus unitate diſcrepant, ſuntque omnes per 6 diviſibiles. Cum jam ſit

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{5.5}{4.6} \cdot \frac{7.7}{6.8} \cdot \frac{11.11}{10.12} \cdot \frac{13.13}{12.14} \cdot \&c.,$$

erit, hac expreſſione per illam diviſa,

$$\frac{\pi\sqrt{3}}{2} = \frac{9}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \&c.,$$

 ubi denominatores non ſunt per 6 diviſibiles. Vel erit

$$\frac{\pi}{2\sqrt{3}} = \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{24} \cdot \&c.$$

$$\frac{2\pi}{3\sqrt{3}} = \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{22} \cdot \&c.,$$

quarum hæc per illam diviſa dat

$$\frac{4}{3} = \frac{6}{4} \cdot \frac{6}{8} \cdot \frac{12}{10} \cdot \frac{12}{14} \cdot \frac{18}{16} \cdot \frac{18}{20} \cdot \frac{24}{22} \cdot \&c.,$$

ſeu

$$\frac{4}{3} = \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{9}{8} \cdot \frac{9}{10} \cdot \frac{12}{11} \cdot \&c.,$$

ubi ſingulæ fractiones ex numeris primis 5, 7, 11, &c., formantur, ſingulos numeros primos in duas partes unitate differentes diſpeſcendo, & partes per 3 diviſibiles conſtanter pro numeratoribus ſumendo.

294. Quoniam vero ſupra vidimus eſſe

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \&c.,$$

ſeu

$$\frac{\pi}{3} = \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \&c.,$$

ſi ſuperiores $\frac{\pi}{2\sqrt{3}}$ & $\frac{2\pi}{3\sqrt{3}}$ per hanc dividantur, orietur

$$\frac{\sqrt{3}}{2} = \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{14}{15} \cdot \frac{16}{15} \cdot \&c.$$

$$\frac{2}{\sqrt{3}} = \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{18}{19} \cdot \frac{24}{23} \cdot \frac{30}{29} \cdot \&c.,$$

In priori expreſſione fractiones formantur ex numeris primis formæ $12m + 6 \pm 1$, in poſteriore ex numeris primis formæ $12m \pm 1$, ſingulos in duas partes unitate diſcrepantes diſpeſcendo, & partes pares pro numeratoribus, impares vero pro denominatoribus ſumendo.

295. Contemplemur adhuc Seriem ſupra inventam (179), quæ ita progrediebatur CAP. XV.

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \&c. = A\text{, erit}$$

$$\frac{1}{3}A = \frac{1}{3} + \frac{1}{9} - \frac{1}{15} - \frac{1}{21} + \frac{1}{27} + \frac{1}{33} - \&c.\text{: ſubtrahatur}$$

$$\left(1 - \frac{1}{3}\right)A = 1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} + \frac{1}{19} - \&c. = B$$

$$\frac{1}{5}B = \frac{1}{5} - \frac{1}{25} - \frac{1}{35} + \frac{1}{55} - \&c.\text{: addatur, erit}$$

$$\left(1 + \frac{1}{5}\right)B = 1 - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17}\ \&c. = C:$$

ſicque progrediendo tandem pervenietur ad

$$\frac{\pi}{2\sqrt{2}}\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7}\right)\left(1 - \frac{1}{11}\right)\left(1 + \frac{1}{13}\right)\left(1 - \frac{1}{17}\right)\left(1 - \frac{1}{19}\right)\&c. = 1,$$

ubi ſigna ita ſe habent, ut numerorum primorum formæ $8m+1$, vel $8m+3$, ſigna ſint —; numerorum primorum vero formæ $8m+5$, vel $8m+7$, ſigna ſint +. Hinc itaque erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\frac{23}{24}\cdot\&c.,$$

ubi omnes denominatores vel diviſibiles ſunt per 8, vel tantum ſunt numeri impariter pares. Cum igitur ſit

$$\frac{\pi}{4} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{24}\cdot\&c.$$

$$\frac{\pi}{2} = \frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{22}\cdot\&c.,$$

&

LIB. I. $\frac{\pi\pi}{8} = \frac{3.3}{2.4} \cdot \frac{5.5}{4.6} \cdot \frac{7.7}{6.8} \cdot \frac{11.11}{10.12} \cdot \frac{13.13}{12.14} \cdot$ &c.,

erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \frac{23}{22} \cdot \text{\&c.},$$

ubi nulli denominatores per 8 diviſibiles occurrunt, pariter pares vero adſunt, quoties unitate differunt a numeratoribus. Prima vero per ultimam diviſa dat

$$1 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{9}{8} \cdot \frac{10}{9} \cdot \frac{11}{12} \cdot \text{\&c.},$$

quæ fractiones formantur ex numeris primis, ſingulos in duas partes unitate diſcrepantes diſpeſcendo, & partes pares (niſi ſint pariter pares) pro numeratoribus ſumendo.

296. Simili modo reliquæ Series, quas ſupra pro expreſſione arcuum circularium invenimus (179 & *ſeqq.*) in Factores transformari poſſunt, qui ex numeris primis conſtituantur. Sicque multæ aliæ inſignes proprietates tam hujuſmodi Factorum, quam Serierum infinitarum erui poterunt. Quoniam vero præcipuas hic jam commemoravi, pluribus evolvendis hic non immorabor. Sed ad aliud huic affine argumentum procedam. Quemadmodum ſcilicet in hoc Capite numeri, quatenus per multiplicationem oriuntur, ſunt conſiderati, ita in ſequenti generatio numerorum per additionem perpendetur.

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CAPUT XVI.

De Partitione numerorum.

297. Propoſita ſit iſta expreſſio

$(1+x^{\alpha}z)(1+x^{\beta}z)(1+x^{\gamma}z)(1+x^{\delta}z)(1+x^{\epsilon}z)$ &c., quæ cujuſmodi induat formam, ſi per multiplicationem evolvatur, inquiramus. Ponamus prodire

$$1+Pz+Qz^{2}+Rz^{3}+Sz^{4}+\&c.,$$

atque manifeſtum eſt P fore ſummam Poteſtatum $x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+x^{\epsilon}+$ &c.. Deinde Q eſt ſumma factorum ex binis Poteſtatibus diverſis, ſeu Q erit aggregatum plurium Poteſtatum ipſius x, quarum Exponentes ſunt ſummæ duorum terminorum diverſorum hujus Seriei

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,\ \zeta,\ \eta,\ \&c.$$

Simili modo R erit aggregatum Poteſtatum ipſius x, quarum Exponentes ſunt ſummæ trium terminorum diverſorum. Atque S erit aggregatum Poteſtatum ipſius x, quarum Exponentes ſunt ſummæ quatuor terminorum diverſorum ejuſdem Seriei, α, β, γ, δ, ϵ, &c., & ita porro.

298. Singulæ hæ Poteſtates ipſius x, quæ in valoribus litterarum P, Q, R, S, &c., inſunt, unitatem pro coëfficiente habebunt; ſi quidem earum Exponentes unico modo ex α, β, γ, δ, &c., formari queant: ſin autem ejuſdem Poteſtatis Exponens pluribus modis poſſit eſſe ſumma duorum,

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 trium, pluriumve terminorum Seriei $\alpha, \beta, \gamma, \delta, \epsilon$, &c., tum etiam Poteſtas illa coëfficientem habebit, qui unitatem toties in ſe complectatur. Sic, ſi in valore ipſius Q reperiatur Nx^n, indicio hoc erit numerum n eſſe N diverſis modis ſummam duorum terminorum diverſorum Seriei α, β, γ, &c.. Atque ſi in evolutione Factorum propoſitorum occurrat terminus $Nx^n z^m$, ejus coëfficiens N indicabit quot variis modis numerus n poſſit eſſe ſumma m terminorum diverſorum Seriei $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, &c.

299. Quod ſi ergo productum propoſitum

$$(1 + x^\alpha z)(1 + x^\beta z)(1 + x^\gamma z)(1 + x^\delta z) \&c.,$$

per multiplicationem veram evolvatur, ex expreſſione reſultante ſtatim apparebit, quot variis modis datus numerus poſſit eſſe ſumma tot terminorum diverſorum Seriei $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, &c., quot quis voluerit. Scilicet, ſi quæratur quot variis modis numerus n poſſit eſſe ſumma m terminorum illius Seriei diverſorum, in expreſſione evoluta quæri debet terminus $x^n z^m$, ejuſque coëfficiens indicabit numerum quæſitum.

300. Quo hæc fiant planiora, ſit propoſitum hoc productum ex Factoribus conſtans infinitis

$$(1 + xz)(1 + x^2 z)(1 + x^3 z)(1 + x^4 z)(1 + x^5 z) \&c.,$$

quod per multiplicationem actualem evolutum dat

$$\begin{aligned}
1+z\,&(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\&c.)\\
+z^2&(x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+4x^{10}+5x^{11}+\&c.)\\
+z^3&(x^6+x^7+2x^8+3x^9+4x^{10}+5x^{11}+7x^{12}+8x^{13}+10x^{14}+\&c.)\\
+z^4&(x^{10}+x^{11}+2x^{12}+3x^{13}+5x^{14}+6x^{15}+9x^{16}+11x^{17}+15x^{18}+\&c.)\\
+z^5&(x^{15}+x^{16}+2x^{17}+3x^{18}+5x^{19}+7x^{20}+10x^{21}+13x^{22}+18x^{23}+\&c.)\\
+z^6&(x^{21}+x^{22}+2x^{23}+3x^{24}+5x^{25}+7x^{26}+11x^{27}+14x^{28}+20x^{29}+\&c.)\\
+z^7&(x^{28}+x^{29}+2x^{30}+3x^{31}+5x^{32}+7x^{33}+11x^{34}+15x^{35}+21x^{36}+\&c.)\\
+z^8&(x^{36}+x^{37}+2x^{38}+3x^{39}+5x^{40}+7x^{41}+11x^{42}+15x^{43}+22x^{44}+\&c.)\\
&\&c.
\end{aligned}$$

Ex his ergo Seriebus ſtatim definire licet quot variis modis propoſitus numerus ex dato terminorum diverſorum hujus Seriei 1, 2, 3, 4, 5, 6, 7, 8, &c. numero oriri queat. Sic, ſi quæratur quot variis modis numerus 35 poſſit eſſe ſumma ſeptem terminorum diverſorum Seriei 1, 2, 3, 4, 5, 6, 7, &c., quæratur in Serie z^7 multiplicante Poteſtatem x^{35}, ejuſque coëfficiens 15 indicabit numerum propoſitum 35 quindecim variis modis eſſe ſummam ſeptem terminorum Seriei 1, 2, 3, 4, 5, 6, 7, 8, &c.

301. Quod ſi autem ponatur $z=1$, & ſimiles Poteſtates ipſius x in unam ſummam conjiciantur, ſeu, quod eodem redit, ſi evolvatur hæc expreſſio infinita

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\&c.,$$

quo facto orietur hæc Series

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+\&c.,$$

ubi quivis coëfficiens indicat, quot variis modis Exponens Poteſtatis ipſius x conjunctæ ex terminis diverſis Seriei 1, 2, 3, 4, 5, 6, 7, &c., per additionem emergere poſſit. Sic apparet numerum 8 ſex modis per additionem diverſorum numerorum produci, qui ſunt

$8=8$ $\qquad$ $8=5+3$
$8=7+1$ $\qquad$ $8=5+2+1$
$8=6+2$ $\qquad$ $8=4+3+1$

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ubi notandum eſt numerum propoſitum ipſum ſimul computari debere, quia numerus terminorum non definitur, ideoque unitas inde non excluditur.

302. Hinc igitur intelligitur, quomodo quiſque numerus per additionem diverſorum numerorum producatur. Conditio autem diverſitatis omittetur, ſi Factores illos in denominatorem transponamus. Sit igitur propoſita hæc expreſſio

$$\frac{1}{(1-x^{\alpha}z)(1-x^{\beta}z)(1-x^{\gamma}z)(1-x^{\delta}z)(1-x^{\epsilon}z)\,\&c.},$$

quæ per diviſionem evoluta det

$$1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c..$$

Atque manifeſtum eſt fore P aggregatum Poteſtatum ipſius x, quarum Exponentes contineantur in hac Serie

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,\ \zeta,\ \eta,\ \&c..$$

Deinde Q erit aggregatum Poteſtatum ipſius x, quarum Exponentes ſint ſummæ duorum terminorum hujus Seriei, ſive eorundem ſive diverſorum. Tum erit R ſumma Poteſtatum ipſius x, quarum Exponentes ex additione trium terminorum illius Seriei oriantur; & S ſumma Poteſtatum, quarum Exponentes ex additione quatuor terminorum in illa Serie contentorum formantur, & ita porro.

303. Si igitur tota expreſſio per ſingulos terminos explicetur, & termini ſimiles conjunctim exprimantur, intelligetur quot variis modis propoſitus numerus n per additionem m terminorum, ſive diverſorum ſive non diverſorum, Seriei α, β, γ, δ, ϵ, ζ, &c., produci queat. Quæratur ſcilicet in expreſſione evoluta terminus $x^n z^m$, ejuſque coëfficiens, qui ſit N, ita ut totus terminus ſit $= Nx^n z^m$, atque coëfficiens N indicabit quot variis modis numerus n per additionem m terminorum

minorum in Serie α, β, γ, δ, ϵ, &c., contentorum produci queat. Hoc igitur pacto quæſtio priori, quam ante ſumus contemplati, ſimilis reſolvetur. CAP. XVI.

304. Accommodemus hæc ad caſum inprimis notatu dignum, ſitque propoſita hæc expreſſio

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\,\&c.},$$

quæ per diviſionem evoluta dabit

$$\begin{aligned}
1&+z\,(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\&c.)\\
&+z^2(x^2+x^3+2x^4+2x^5+3x^6+3x^7+4x^8+4x^9+5x^{10}+\&c.)\\
&+z^3(x^3+x^4+2x^5+3x^6+4x^7+5x^8+7x^9+8x^{10}+10x^{11}+\&c.)\\
&+z^4(x^4+x^5+2x^6+3x^7+5x^8+6x^9+9x^{10}+11x^{11}+15x^{12}+\&c.)\\
&+z^5(x^5+x^6+2x^7+3x^8+5x^9+7x^{10}+10x^{11}+13x^{12}+18x^{13}+\&c.)\\
&+z^6(x^6+x^7+2x^8+3x^9+5x^{10}+7x^{11}+11x^{12}+14x^{13}+20x^{14}+\&c.)\\
&+z^7(x^7+x^8+2x^9+3x^{10}+5x^{11}+7x^{12}+11x^{13}+15x^{14}+21x^{15}+\&c.)\\
&+z^8(x^8+x^9+2x^{10}+3x^{11}+5x^{12}+7x^{13}+11x^{14}+15x^{15}+22x^{16}+\&c.)\\
&\&c.,
\end{aligned}$$

Ex his ergo Seriebus ſtatim definire licet quot variis modis propoſitus numerus per additionem ex dato terminorum hujus Seriei 1, 2, 3, 4, 5, 6, 7, &c., numero produci queat. Sic, ſi quæratur quot variis modis numerus 13 oriri poſſit per additionem quinque numerorum integrorum, ſpectari debebit terminus $x^{13}z^5$, cujus coëfficiens 18 indicat numerum propoſitum 13 ex quinque numerorum additione octodecim modis oriri poſſe.

305. Si ponatur $z = 1$, atque ſimiles Poteſtates ipſius x conjunctim exprimantur, hæc expreſſio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\,\&c.},$$

evolvetur in hanc Seriem

LIB. I. $1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8$ &c.,

in qua quilibet coëfficiens indicat, quot variis modis Exponens Poteſtatis adjunctæ per additionem produci queat ex numeris integris, ſive æqualibus ſive inæqualibus. Scilicet ex termino $11x^6$ cognoſcitur numerum 6 undecim modis per additionem numerorum integrorum produci poſſe, qui ſunt

$6=6$	$6=3+1+1+1$
$6=5+1$	$6=2+2+2$
$6=4+2$	$6=2+2+1+1$
$6=4+1+1$	$6=2+1+1+1+1$
$6=3+3$	$6=1+1+1+1+1+1$
$6=3+2+1$	

ubi quoque notari debet, ipſum numerum propoſitum, cum in Serie numerorum $1, 2, 3, 4, 5, 6$, &c., propoſita contineatur, unum modum præbere.

306. His in genere expoſitis, diligentius inquiramus in modum hanc compoſitionum multitudinem inveniendi. Ac primo quidem conſideremus eam ex numeris integris compoſitionem, in qua numeri tantum diverſi admittuntur, quam prius commemoravimus. Sit igitur in hunc finem propoſita hæc expreſſio

$$Z=(1+xz)(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)\ \&c.,$$

quæ evoluta & ſecundum Poteſtates ipſius z digeſta præbeat

$$Z=1+Pz+Qz^2+Rz^3+Sz^4+Tz^5+\&c.,$$

ubi methodus deſideratur has ipſius x Functiones P, Q, R, S, T, &c., expedite inveniendi, hoc enim pacto quæſtioni propoſitæ convenientiſſime ſatisfiet.

307. Patet autem, ſi loco z ponatur xz, prodire $(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)$ &c. $= \frac{Z}{1+xz}$ ergo, poſito xz loco z, valor producti, qui erat Z, abibit in $\frac{Z}{1+xz}$; ſicque, cum ſit

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c.,$$

erit

$$\frac{Z}{1+xz} = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c.;$$

multiplicetur ergo actu per $1+xz$, atque prodibit

$$\begin{aligned} Z = 1 &+ Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c. \\ &+ xz + Px^2z^2 + Qx^3z^3 + Rx^4z^4 + \&c., \end{aligned}$$

qui valor ipſius Z cum ſuperiori comparatus dabit

$$P = \frac{x}{1-x};\ Q = \frac{Px^2}{1-x^2};\ R = \frac{Qx^3}{1-x^3};\ S = \frac{Rx^4}{1-x^4}\ \&c.,$$

Sequentes ergo pro P, Q, R, S, &c., obtinentur valores

$$\begin{aligned} P &= \frac{x}{1-x} \\ Q &= \frac{x^3}{(1-x)(1-x^2)} \\ R &= \frac{x^6}{(1-x)(1-x^2)(1-x^3)} \\ S &= \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} \\ T &= \frac{x^{15}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)} \\ &\&c. \end{aligned}$$

308. Sic igitur ſeorſim unamquamque Seriem Poteſtatum ipſius x exhibere poſſumus, ex qua definire licet, quot variis modis propoſitus numerus ex dato partium integrarum numero per additionem formari poſſit. Manifeſtum autem porro eſt has ſingulas Series eſſe recurrentes, quia ex evolutione

LIB. I. Functionis fractæ ipſius x naſcuntur. Prima ſcilicet expreſſio $P = \frac{x}{1-x}$, dat Seriem geometricam

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c.,$$

ex qua quidem manifeſtum eſt quemvis numerum ſemel in Serie numerorum integrorum contineri.

309. Expreſſio ſecunda $\frac{x^3}{(1-x)(1-x^2)}$, dat hanc Seriem $x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + \&c.$, in qua cujuſvis termini coëfficiens indicat quot modis Exponens ipſius x in duas partes inæquales diſpertiri poſſit. Sic terminus $4x^9$ indicat, numerum 9 quatuor modis in duas partes inæquales ſecari poſſe. Quod ſi hanc Seriem per x^3 dividamus, prodibit Series, quam præbet iſta fractio $\frac{1}{(1-x)(1-x^2)}$, quæ erit

$$1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + \&c.,$$

cujus terminus generalis ſit $= Nx^n$; atque ex geneſi hujus Seriei intelligitur coëfficientem N indicare, quot variis modis Exponens n ex numeris 1 & 2 per additionem naſci queat. Cum igitur prioris Seriei terminus generalis ſit $= Nx^{n+3}$, deducitur hinc iſtud theorema.

Quot variis modis numerus n *per additionem ex numeris* 1 & 2 *produci poteſt, totidem variis modis numerus* $n + 3$ *in duas partes inæquales ſecari poterit.*

310. Expreſſio tertia $\frac{x^6}{(1-x)(1-x^2)(1-x^3)}$ in Seriem evoluta dabit

$$x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + \&c.,$$

in qua cujuſvis termini coëfficiens indicat quot variis modis Exponens Poteſtatis x adjunctæ in tres partes inæquales diſper-

tiri poſſit. Quod ſi autem hæc fractio $\frac{1}{(1-x)(1-x^2)(1-x^3)}$ evolvatur, prodibit hæc Series

$$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + \&c.,$$

cujus terminus generalis ſi ponatur $= Nx^n$, coëfficiens N indicabit quot variis modis numerus n ex numeris 1, 2, 3, per additionem produci poſſit. Cum igitur prioris Seriei terminus generalis ſit Nx^{n+6}, ſequetur hinc iſtud theorema.

Quot variis modis numerus n *per additionem ex numeris* 1, 2, 3, *produci poteſt, totidem variis modis numerus* n + 6 *in tres partes inæquales ſecari poterit.*

311. Expreſſio quarta $\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$ in Seriem recurrentem evoluta dabit

$$x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + \&c.;$$

in qua cujuſvis termini coëfficiens indicabit quot variis modis Exponens Poteſtatis x adjunctæ in quatuor partes inæquales diſpertiri poſſit. Quod ſi autem hæc expreſſio $\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$ evolvatur, prodibit ſuperior Series per x^{10} diviſa, nempe

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + 11x^7 + \&c.;$$

cujus terminum generalem ponamus $= Nx^n$; atque hinc patebit coëfficientem N indicare, quot variis modis numerus n per additionem oriri poſſit ex his quatuor numeris 1, 2, 3, 4. Cum igitur prioris Seriei terminus generalis futurus ſit $= Nx^{n+10}$, deducitur hoc theorema.

Quot variis modis numerus n *per additionem produci poteſt*

LIB. I. *ex numeris* 1, 2, 3, 4, *totidem variis modis numerus* n + 10 *in quatuor partes inæquales ſecari poterit.*

312. Generaliter ergo, ſi hæc expreſſio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

in Seriem evolvatur, ejuſque terminus generalis fuerit $= Nx^n$, coëfficiens N indicabit, quot variis modis numerus n per additionem produci poſſit ex his numeris 1, 2, 3, 4 .... m. Quod ſi autem hæc expreſſio

$$\frac{x^{\frac{m(m+1)}{2}}}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

in Seriem evolvatur, erit ejus terminus generalis $= Nx^{n+\frac{m(m+1)}{2}}$: atque hic coëfficiens N indicat quot variis modis numerus $n + \frac{m(m+1)}{1.\ 2}$ in m partes inæquales ſecari poſſit, unde hoc habetur theorema.

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, 4 .... m, *totidem modis numerus* $\mathrm{n} + \frac{\mathrm{m}(\mathrm{m}+1)}{1.\ 2}$ *in* m *partes inæquales ſecari poterit.*

313. Ex poſita partitione numerorum in partes inæquales, perpendamus quoque partitionem in partes, ubi æqualitas partium non excluditur; quæ partitio ex hac expreſſione originem habet

$$Z = \frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\,\&c.}$$

Ponamus evolutione per diviſionem inſtituta prodire

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \&c.$$

Perſpicuum autem eſt, ſi loco z ponatur xz, prodire

$$\frac{1}{(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\ \&c.} = (1-xz)\,Z.$$

Facta ergo in Serie evoluta eadem mutatione, fiet

$$(1-xz)Z = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c.;$$

Multiplicetur ergo ſuperior Series pariter per $(1-xz)$, eritque

$$\begin{aligned}(1-xz)\,Z = 1 &+ Pz + Qz^2 + Rz^3 + Sz^4 + \&c.\\ &- xz - Pxz^2 - Qxz^3 - Rxz^4 - \&c.\end{aligned}$$

Comparatione ergo inſtituta orietur

$$P = \frac{x}{1-x};\ Q = \frac{Px}{1-x^2};\ R = \frac{Qx}{1-x^3};\ S = \frac{Rx}{1-x^4}\ \&c.;$$

unde pro P, Q, R, S, &c., ſequentes valores proveniunt.

$$P = \frac{x}{1-x}$$

$$Q = \frac{x^2}{(1-x)(1-x^2)}$$

$$R = \frac{x^3}{(1-x)(1-x^2)(1-x^3)}$$

$$S = \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

&c.

314. Expreſſiones iſtæ a ſuperioribus aliter non diſcrepant, niſi quod numeratores hic minores habeant Exponentes quam caſu præcedente. Atque hanc ob rem Series, quæ per evolutionem naſcuntur, ratione coëfficientium omnino convenient, quæ convenientia jam ex comparatione (§. §. 300 & 304.) perſpicitur, nunc vero demum ejus ratio intelligitur. Hinc ergo omnino ſimilia theoremata conſequentur, quæ ſunt.

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LIB. I. *Quot variis modis numerus* n *per additionem produci poteſt ex numeris* 1, 2, *totidem modis numerus* n + 2 *in duas partes diſpertiri poterit.*

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3; *totidem modis numerus* n + 3 *in tres partes diſpertiri poterit.*

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, 4, *totidem modis numerus* n + 4 *in quatuor partes diſpertiri poterit.*

Atque generaliter habebitur hoc theorema :

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, ...m, *totidem modis numerus* n + m *in* m *partes diſpertiri poterit.*

315. Sive ergo quæratur quot modis datus numerus in m partes inæquales, ſive in m partes, æqualibus non excluſis, diſpertiri poſſit, utraque quæſtio reſolvetur ſi cognoſcatur quot modis quiſque numerus per additionem produci poſſit ex numeris 1, 2, 3, 4 .... m, quemadmodum hoc patebit ex ſequentibus theorematibus, quæ ex ſuperioribus ſunt derivata.

Numerus n *tot modis in* m *partes inæquales diſpertiri poteſt, quot modis numerus* $\mathrm{n} - \frac{\mathrm{m}(\mathrm{m}+1)}{2}$ *per additionem produci poteſt ex numeris* 1, 2, 3, 4, ... m.

Numerus n, *tot modis in* m *partes ſive æquales ſive inæquales diſpertiri poteſt quot modis numerus* n — m *per additionem produci poteſt ex numeris* 1, 2, 3, ... m.

Hinc porro ſequuntur hæc theoremata.

Numerus n *totidem modis in* m *partes inæquales ſecari poteſt, quot modis numerus* $\mathrm{n} - \frac{\mathrm{m}(\mathrm{m}-1)}{2}$ *in* m *partes, ſive æquales ſive inæquales, diſpertitur.*

Numerus

Numerus n *totidem modis in* m *partes, ſive inæquales ſive æquales, ſecari poteſt, quot modis numerus* $n+\frac{m(m-1)}{2}$ *in* m *partes inæquales diſpertiri poteſt.* CAP. XVI.

316. Per formationem autem Serierum recurrentium inveniri poterit, quot variis modis datus numerus n per additionem produci poſſit ex numeris $1, 2, 3, \ldots\ldots\ldots m$. Ad hoc enim inveniendum evolvi debebit fractio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

atque Series recurrens continuari debebit uſque ad terminum Nx^n, cujus coëfficiens N indicabit, quot modis numerus n per additionem produci poſſit ex numeris $1, 2, 3, 4, \ldots\ldots m$. At vero hic ſolvendi modus non parum habebit difficultatis, ſi numeri m & n ſint modice magni; ſcala enim relationis, quam præbet denominator per multiplicationem evolutus, ex pluribus terminis conſtat, unde operoſum erit Seriem ad plures terminos continuare.

317. Hæc autem diſquiſitio minus erit moleſta, ſi caſus ſimpliciores primum expediantur, ex his enim facile erit ad caſus magis compoſitos progredi. Sit Seriei, quæ ex hac fractione oritur,

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

terminus generalis $= Nx^n$; at Seriei ex hac forma

$$\frac{x^m}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

ortæ terminus generalis ſit Mx^n, ubi coëfficiens M indica-

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LIB. I. bit quot variis modis numerus $n - m$ per additionem produci poſſit ex numeris 1, 2, 3, ... m. Subtrahatur poſterior expreſſio a priori, ac remanebit

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^{m-1})}$$

atque manifeſtum eſt Seriei hinc ortæ terminum generalem futurum eſſe $(N - M)\, x^n$; quare coëfficiens $N - M$ indicabit quot variis modis numerus n per additionem produci poſſit ex numeris 1, 2, 3, ... $(m - 1)$.

318. Hinc ergo ſequentem regulam nanciſcimur.

Sit L numerus modorum, quibus numerus n per additionem produci poteſt ex numeris 1, 2, 3, ... $(m - 1)$.

Sit M numerus modorum, quibus numerus $n - m$ per additionem produci poteſt ex numeris 1, 2, 3, ... m.

Sitque N numerus modorum, quibus numerus n per additionem produci poteſt ex numeris 1, 2, 3, ... m.

His poſitis, erit, ut vidimus, $L = N - M$; ideoque $N = L + M$. Quod ſi ergo jam invenerimus quot variis modis numeri n & $n - m$ per additionem produci queant, ille ex numeris 1, 2, 3, ... $(m - 1)$, hic vero ex numeris 1, 2, 3, ... m; hinc addendo cognoſcemus, quot variis modis numerus n per additionem produci queat ex numeris 1, 2, 3, ... m. Ope hujus theorematis a caſibus ſimplicioribus, qui nihil habent difficultatis, continuo ad magis compoſitos progredi licebit, hocque modo tabula hic annexa eſt computata, * cujus uſus ita ſe habet.

Si quæratur quot variis modis numerus 50 in 7 partes inæquales diſpertiri poſſit; ſumatur in prima columna verticali numerus $50 - \frac{7 \cdot 8}{2} = 22$, in horizontali autem ſuprema numerus romanus VII; atque numerus in angulo poſitus 522 indicabit modorum numerum quæſitum.

* *Vide Tab. pag.* 275.

Sin autem quæratur, quot variis modis numerus 50 in 7 partes, ſive æquales ſive inæquales, diſpertiri poſſit, in prima columna verticali ſumatur numerus $50 - 7 = 43$, cui in columna 7ma reſpondebit numerus quæſitus 8946.

319. Series hujus tabulæ verticales, etſi ſunt recurrentes, tamen ingentem habent connexionem cum numeris naturalibus, trigonalibus, pyramidalibus & ſequentibus, quam paucis exponere operæ pretium erit. Quoniam enim ex fractione $\frac{1}{(1-x)(1-xx)}$ oritur Series $1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 +$ &c., ac proinde ex fractione $\frac{1}{(1-x)(1-xx)}$ hæc $x + x^2 + 2x^3 + 2x^4 + 3x^5 + 3x^6 +$ &c.. Si duæ hæ Series addantur, naſcitur iſta

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \&c.,$$

quæ per diviſionem oritur ex fractione $\frac{1+x}{(1-x)(1-xx)} = \frac{1}{(1-x)^2}$; unde patet Seriei poſtremæ terminos numericos Seriem numerorum naturalium conſtituere. Hinc ex Serie tabulæ ſecunda addendo binos terminos proveniet Series numerorum naturalium, poſito $x = 1$.

$$1+1+2+2+3+3+4+4+5+5+6+6+\&c.$$
$$1+2+3+4+5+6+7+8+9+10+11+12+\&c.$$

Viciſſim ergo ex Serie numerorum naturalium ſuperior invenitur, ſubtrahendo quemque terminum Seriei ſuperioris a termino inferioris ſequente.

320. Series verticalis tertia oritur ex fractione $\frac{1}{(1-x)(1-xx)(1-x^3)}$. Cum autem ſit $\frac{1}{(1-x)^3} = \frac{(1+x)(1+x+xx)}{(1-x)(1-xx)(1-x^3)}$, manifeſtum eſt, ſi primo Seriei

 illius terni termini addantur, tum bini hujus novæ Seriei denuo addantur, prodire debere numeros trigonales, id quod ex ſchemate ſequente apparebit.

$$1+1+2+3+4+5+7+8+10+12+14+16+19\ \&c.$$
$$1+2+4+6+9+12+16+20+25+30+36+42+49\ \&c.$$
$$1+3+6+10+15+21+28+36+45+55+66+78+91\ \&c.$$

Viciſſim autem apparet quomodo ex Serie trigonalium erui debeat Series ſuperior.

321. Simili modo, quia Series quarta oritur ex fractione $\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)}$, erit $\frac{(1+x)(1+x+xx)(1+x+xx+x^3)}{(1-x)(1-xx)(1-x^3)(1-x^4)} = \frac{1}{(1-x)^4}$. Si in Serie quarta primum quaterni termini addantur, tum in Serie reſultante terni, denique in hac bini, prodibit Series numerorum pyramidalium uti ex ſequenti calculo videre licet.

$$1+1+2+3+5+6+9+11+15+18+23+27+\&c.$$
$$1+2+4+7+11+16+23+31+41+53+67+83+\&c.$$
$$1+3+7+13+22+34+50+70+95+125+161+203+\&c.$$
$$1+4+10+20+35+56+84+120+165+220+286+364+\&c.$$

Simili autem modo Series quinta deducet ad numeros pyramidales ſecundi ordinis, ſexta ad tertii ordinis & ita porro.

322. Viciſſim igitur ex numeris figuratis illæ ipſæ Series, quæ in tabulis occurrunt, formari poterunt, per operationes, quæ ex inſpectione calculi ſequentis ſponte elucebunt.

$$\begin{array}{l}
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + \&c. \\
1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 5 + 5 + \&c. \quad \text{II}
\end{array}$$

$$\begin{array}{l}
1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + \&c. \\
1 + 2 + 4 + 6 + 9 + 12 + 16 + 20 + 25 + 30 + \&c. \\
1 + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 12 + \&c. \quad \text{III}
\end{array}$$

$$\begin{array}{l}
1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + 165 + 220 + \&c. \\
1 + 3 + 7 + 13 + 22 + 34 + 50 + 70 + 95 + 125 + \&c. \\
1 + 2 + 4 + 7 + 11 + 16 + 23 + 31 + 41 + 53 + \&c. \\
1 + 1 + 2 + 3 + 5 + 6 + 9 + 11 + 15 + 18 + \&c. \quad \text{IV}
\end{array}$$

$$\begin{array}{l}
1 + 5 + 15 + 35 + 70 + 126 + 210 + 330 + 495 + 715 + \&c. \\
1 + 4 + 11 + 24 + 46 + 80 + 130 + 200 + 295 + 420 + \&c. \\
1 + 3 + 7 + 14 + 25 + 41 + 64 + 95 + 136 + 189 + \&c. \\
1 + 2 + 4 + 7 + 12 + 18 + 27 + 38 + 53 + 71 + \&c. \\
1 + 1 + 2 + 3 + 5 + 7 + 10 + 13 + 18 + 23 + \&c. \quad \text{V}
\end{array}$$

&c.

In his ordinibus primæ Series ſunt numeri figurati, unde ſubtrahendo quemvis terminum Seriei ſecundæ a termino primæ ſequente formatur Series ſecunda. Tum Seriei tertiæ bini termini conjunctim ſubtrahantur a termino ſequente Seriei ſecundæ, ſicque oritur Series tertia; hocque modo ſubtrahendo ulterius ſummam trium, quatuor, & ita porro terminorum a termino ſuperioris Seriei ſequente, formabuntur reliquæ Series donec perveniatur ad Seriem, quæ incipit ab $1 + 1 + 2$ &c., hæcque erit Series in tabula exhibita.

323. Series verticales tabulæ omnes ſimiliter incipiunt, continuoque plures habent terminos communes; ex quo intelligitur in infinitum has Series inter ſe fore congruentes. Prodibit autem Series, quæ oritur ex hac fractione

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\,\&c.}$$

quæ cum ſit recurrens, primum denominator ſpectari debet, ut

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 hinc ſcala relationis habeatur. Quod ſi autem Factores denominatoris continuo in ſe multiplicentur, prodibit

$$1 - x - x^{2} + x^{5} + x^{7} - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \&c.,$$

quæ Series ſi attentius conſideretur, aliæ Poteſtates ipſius x adeſſe non deprehenduntur, niſi quarum Exponentes contineantur in hac formula $\frac{3nn+n}{2}$; atque, ſi n ſit numerus impar, Poteſtates erūnt negativæ; affirmativæ autem ſi n fuerit numerus par.

324. Cum igitur ſcala relationis ſit

$$+1, +1, 0, 0, -1, 0, -1, 0, 0, 0, 0, +1, 0, 0, +1, 0, 0, \&c.$$

Series recurrens ex evolutione fractionis

$$\frac{1}{(1-x)(1-x^{2})(1-x^{3})(1-x^{4})(1-x^{5})(1-x^{6})(1-x^{7})\ \&c.},$$

oriunda erit hæc

$$1 + x + 2x^{2} + 3x^{3} + 5x^{4} + 7x^{5} + 11x^{6} + 15x^{7} + 22x^{8} + 30x^{9} + 42x^{10} + 56x^{11} + 77x^{12} + 101x^{13} + 135x^{14} + 176x^{15} + 231x^{16} + 297x^{17} + 385x^{18} + 490x^{19} + 627x^{20} + 792x^{21} + 1002x^{22} + 1250x^{23} + 1570x^{24}\ \&c.$$

In hac ergo Serie coëfficiens quiſque indicat, quot variis modis Exponens ipſius x per additionem ex numeris integris oriri queat. Sic numerus 7 quindecim modis per additionem oriri poteſt.

$7=7$	$7=4+2+1$	$7=3+1+1+1+1$
$7=6+1$	$7=4+1+1+1$	$7=2+2+2+1$
$7=5+2$	$7=3+3+1$	$7=2+2+1+1+1$
$7=5+1+1$	$7=3+2+2$	$7=2+1+1+1+1+1$
$7=4+3$	$7=3+2+1+1$	$7=1+1+1+1+1+1+1$

325. Quod ſi autem hoc productum

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\text{ \&c.},$$

evolvatur, ſequens prodibit Series

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+\text{ \&c.},$$

in qua quiſque coëfficiens indicat, quot variis modis Exponens ipſius x per additionem numerorum inæqualium oriri poſſit. Sic numerus 9 octo variis modis per additionem ex numeris inæqualibus formari poteſt.

$9=9$	$9=6+2+1$
$9=8+1$	$9=5+4$
$9=7+2$	$9=5+3+1$
$9=6+3$	$9=4+3+2$

326. Ut comparationem inter has formas inſtituamus, ſit

$$P=(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\text{ \&c.},$$

&

$$Q=(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\text{ \&c.},$$

erit

$$PQ=(1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\text{ \&c.},$$

qui Factores cum omnes in P contineantur, dividatur P per PQ, erit $\frac{1}{Q}=(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)$ &c., ideoque

$$Q=\frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\text{ \&c.}},$$

quæ fractio ſi evolvatur, prodibit Series, in qua quiſque coëfficiens indicabit, quot variis modis Exponens ipſius x, per additionem ex numeris imparibus produci poſſit. Cum igitur hæc expreſſio æqualis ſit illi, quam in §. præcedente contemplati ſumus, ſequitur hinc iſtud theorema.

LIB. I. *Quot modis datus numerus per additionem formari poteſt ex omnibus numeris integris inter ſe inæqualibus ; totidem modis idem numerus formari poterit per additionem ex numeris tantum imparibus , ſive æqualibus ſive inæqualibus.*

327. Cum igitur , ut ante vidimus , ſit

$$P = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \&c.,$$

erit , ſcribendo xx loco x ,

$$PQ = 1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + x^{44} + x^{52} - \&c..$$

Quocirca erit hanc per illam dividendo

$$Q = \frac{1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + \&c.}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \&c.}.$$

Erit ergo Series Q pariter recurrens , atque ex Serie $\frac{1}{P}$ oritur , hanc per $1 - x^2 - x^4 + x^{10} + x^{14} - x^{24}$ &c. multiplicando. Nempe , cum ſit (324) $\frac{1}{P} = 1 + x + 2x^2 +$

$$3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \&c.,$$

ſi is multiplicetur per

$$1 - x^2 - x^4 + x^{10} + x^{14} - \&c.,$$

fiet

$$\begin{aligned} &1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \&c. \\ &\quad - x^2 - x^3 - 2x^4 - 3x^5 - 5x^6 - 7x^7 - 11x^8 - 15x^9 - \&c. \\ &\quad\quad - x^4 - x^5 - 2x^6 - 3x^7 - 5x^8 - 7x^9 - \&c. \end{aligned}$$

aut

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \&c.$$

$= Q$. Hinc ergo , ſi formatio numerorum per additionem numerorum , ſive æqualium ſive inæqualium conſtet , deducetur formatio numerorum per additionem numerorum inæqualium , hincque porro formatio numerorum per additionem numerorum imparium tantum.

328. Reſtant in hoc genere caſus quidam memorabiles , quorum evolutio non omni utilitate carebit in numerorum natura cognoſcenda. Conſideretur nempe hæc expreſſio

$(1 + x$

$(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})$&c.,
in qua Exponentes ipſius x in ratione dupla progrediuntur. Hæc expreſſio ſi evolvatur, reperietur quidem hæc Series

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+\&c.;$$

quoniam vero dubium eſſe poteſt, utrum hæc Series in infinitum hac lege geometrica progrediatur, hanc ipſam Seriem inveſtigemus. Sit igitur

$$P=(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\ \&c.,$$

ac ponatur Series per evolutionem oriunda

$$P=1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\epsilon x^5+\zeta x^6+\eta x^7+\theta x^8+\&c.;$$

Patet autem ſi loco x ſcribatur xx, tum prodire productum

$$(1+xx)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})\ \&c.=\frac{P}{1+x}:$$

facta ergo in Serie eadem ſubſtitutione erit

$$\frac{P}{1+x}=1+\alpha x^2+\beta x^4+\gamma x^6+\delta x^8+\epsilon x^{10}+\zeta x^{12}+\&c.;$$

multiplicetur ergo per $1+x$, eritque

$$P=1+x+\alpha x^2+\alpha x^3+\beta x^4+\beta x^5+\gamma x^6+\gamma x^7+\delta x^8+\delta x^9+\&c.,$$

qui valor ipſius P ſi cum ſuperiori comparetur, habebitur

$$\alpha=1;\ \beta=\alpha;\ \gamma=\alpha;\ \delta=\beta;\ \epsilon=\beta;\ \zeta=\gamma;\ \eta=\gamma;\ \&c.,$$

erunt ergo omnes coëfficientes $=1$, ideoque productum propoſitum P evolutum dabit Seriem geometricam

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+\&c..$$

329. Cum igitur hic omnes ipſius x Poteſtates, ſingulæque ſemel occurrant, ex forma producti $(1+x)(1+x^2)(1+x^4)$&c., ſequitur, omnem numerum integrum ex terminis progreſſionis

 geometricæ duplæ 1, 2, 4, 8, 16, 32, &c., diverſis per additionem formari poſſe, hocque unico modo. Nota eſt hæc proprietas in praxi ponderandi, ſi enim habeantur pondera, 1, 2, 4, 8, 16, 32, &c., librarum; his ſolis ponderibus omnia onera ponderari poterunt, niſi partes libræ requirant. Sic his decem ponderibus, nempe 1 ℔, 2 ℔, 4 ℔, 8 ℔, 16 ℔, 32 ℔, 64 ℔, 128 ℔, 256 ℔, 512 ℔, omnia pondera uſque ad 1024 ℔, librari poſſunt, & ſi unum pondus 1024 ℔, addatur omnibus oneribus uſque ad 2048 ℔, ponderandis ſufficient.

330. Oſtendi autem inſuper ſolet in praxi ponderandi paucioribus ponderibus, quæ ſcilicet in ratione geometrica tripla progrediantur, nempe 1, 3, 9, 27, 81, &c., librarum pariter omnia onera ponderari poſſe, niſi opus ſit fractionibus. In hac autem praxi pondera non ſolum uni lanci, ſed ambabus, uti neceſſitas exigit, imponi debent. Nititur ergo iſta praxis hoc fundamento, quod ex terminis progreſſionis geometricæ triplæ 1, 3, 9, 27, 81, &c., diverſis ſemper ſumendis per additionem ac ſubtractionem omnes omnino numeri produci queant; erit ſcilicet.

$1 = 1$	$5 = 9 - 3 - 1$	$9 = 9$
$2 = 3 - 1$	$6 = 9 - 3$	$10 = 9 + 1$
$3 = 3$	$7 = 9 - 3 + 1$	$11 = 9 + 3 - 1$
$4 = 3 + 1$	$8 = 9 - 1$	$12 = 9 + 3$
	&c.	

331. Ad hanc veritatem oſtendendam conſidero hoc productum infinitum

$$(x^{-1}+1+x^{1})(x^{-3}+1+x^{3})(x^{-9}+1+x^{9})(x^{-27}+1+x^{27})\,\&c. = P,$$

quod evolutum alias non dabit Poteſtates ipſius x, niſi quarum Exponentes formari poſſint ex numeris 1, 3, 9, 27, 81, &c.,

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ſive addendo ſive ſubtrahendo : num vero omnes Poteſtates prodeant, ſingulæque ſemel, ſic exploro. Sit

$$P = \&c. + cx^{-3} + bx^{-2} + ax^{-1} + 1 + \alpha x^{1} + \beta x^{2} + \gamma x^{3} + \delta x^{4} + \epsilon x^{5} + \&c.,$$

manifeſtum vero eſt, ſi x^3 loco x ſcribatur, tum prodire

$$\frac{P}{x^{-1} + 1 + x^{1}} = bx^{-6} + ax^{-3} + 1 + \alpha x^{3} + \beta x^{6} + \gamma x^{9} + \&c.;$$

Hinc igitur reperitur $P = \&c.$

$$+ ax^{-4} + ax^{-3} + ax^{-2} + x^{-1} + 1 + x + \alpha x^{2} + \alpha x^{3} + \alpha x^{4} + \beta x^{5} + \beta x^{6} + \beta x^{7} + \&c.,$$

quæ expreſſio cum aſſumta comparata dabit

$$\alpha = 1;\ \beta = \alpha;\ \gamma = \alpha;\ \delta = \alpha;\ \epsilon = \beta;\ \zeta = \beta;\ \&c.,\ \&$$
$$a = 1,\ b = a,\ c = a,\ d = a,\ e = b,\ \&c..$$

Hinc itaque erit

$$P = 1 + x + x^{2} + x^{3} + x^{4} + x^{5} + x^{6} + x^{7} + \&c.$$
$$+ x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5} + x^{-6} + x^{-7} + \&c.,$$

unde patet omnes ipſius x Poteſtates, tam affirmativas quam negativas, hic occurrere, atque adeo omnes numeros ex terminis progreſſionis geometricæ triplæ, vel addendo vel ſubtrahendo, formari poſſe; & unumquemque numerum unico tantum modo.

CAPUT XVII.

De uſu Serierum recurrentium in radicibus æquationum indagandis.

332. INDICAVIT *Vir Celeb. Daniel* BERNOULLI inſignem uſum Serierum recurrentium in inveſtigandis radicibus æquationum cujuſvis gradus, in *Coment. Acad. Petropol. Tomo III*, ubi oſtendit, quemadmodum cujuſque æquationis algebraicæ, quotcunque fuerit dimentionum, valores radicum veris proximi ope Serierum recurrentium aſſignari queant. Quæ inventio, cum ſæpe numero maximam afferat utilitatem, eam hic diligentius explicare conſtitui, ut intelligatur, quibus caſibus adhiberi poſſit. Interdum enim præter expectationem evenit, ut nulla æquationis radix ope hujus methodi cognoſci queat. Quocirca, ut vis hujus methodi clarius perſpiciatur, ex proprietatibus Serierum recurrentium totum fundamentum, quo nititur, contemplemur.

333. Quoniam omnis Series recurrens ex evolutione cujuſdam fractionis rationalis oritur, ſit iſta fractio

$$= \frac{a + bz + cz^2 + dz^3 + ez^4 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.},$$

unde oriatur ſequens Series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus coëfficientes A, B, C, D, &c., ita determinantur ut ſit

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$$A = a$$
$$B = \alpha A + b$$
$$C = \alpha B + \beta A + c$$
$$D = \alpha C + \beta B + \gamma A + d$$
$$E = \alpha D + \beta C + \gamma B + \delta A + e$$
&c.

Terminus autem generalis, ſeu coëfficiens Poteſtatis z^n, invenitur ex reſolutione fractionis propoſitæ in fractiones ſimplices, quarum denominatores ſint Factores denominatoris $1 - \alpha z - \beta zz - \gamma z^3 -$ &c., uti (*Cap. XIII.*) eſt oſtenſum.

334. Forma autem termini generalis potiſſimum pendet ab indole Factorum ſimplicium denominatoris, utrum ſint reales an imaginarii, & utrum ſint inter ſe inæquales & eorum bini pluresve æquales. Quos varios caſus ut ordine percurramus, ponamus primum omnes denominatoris Factores ſimplices cum reales eſſe tum inter ſe inæquales. Sint ergo Factores ſimplices denominatoris omnes $(1-pz)(1-qz)(1-rz)(1-sz)$ &c., ex quibus fractio propoſita in ſequentes fractiones ſimplices reſolvatur $\frac{A}{1-pz} + \frac{B}{1-qz} + \frac{C}{1-rz} + \frac{D}{1-sz} +$ &c.. Quibus cognitis erit Seriei recurrentis terminus generalis $= z^n (Ap^n + Bq^n + Cr^n + Ds^n +$ &c.$)$, quem ſtatuamus $= Pz^n$; ſit ſcilicet P coëfficiens Poteſtatis z^n, ſequentiumque Q, R, &c., ita ut Series recurrens fiat

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + \text{\&c.}$$

335. Ponamus jam n eſſe numerum maximum, ſeu Seriem recurrentem ad plurimos terminos eſſe continuatam; quoniam numerorum inæqualium Poteſtates eo magis fiunt inæquales, quo fuerint altiores; tanta erit diverſitas in Poteſtatibus

 Ap^n, Bq^n, Cr^n, &c., ut ea, quæ oritur ex maximo numerorum p, q, r, &c., reliquas magnitudine longe ſuperet, præ eaque reliquæ penitus evaneſcant, ſi n fuerit numerus plane infinite magnus. Cum igitur numeri p, q, r, &c., ſint inter ſe inæquales, ponamus inter eos p eſſe maximum; ac propterea, ſi n ſit numerus infinitus, fiet $P = Ap^n$; ſin autem n ſit numerus vehementer magnus erit tantum proxime $P = Ap^n$. Simili vero modo erit $Q = Ap^{n+1}$, ideoque $\frac{Q}{P} = p$. Unde patet, ſi Series recurrens jam longe fuerit producta, coëfficientem cujuſque termini per præcedentem diviſum proxime eſſe exhibiturum valorem maximæ litteræ p.

336. Si igitur in fractione propoſita

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

denominator habeat omnes Factores ſimplices reales & inter ſe inæquales, ex Serie recurrente inde orta cognoſci poterit unus Factor ſimplex, is ſcilicet $1 - pz$, in quo littera p omnium maximum habet valorem. Neque in hoc negotio coëfficientes numeratoris a, b, c, d, &c., in computum ingrediuntur, ſed quicunque ii ſtatuantur, tamen denique idem verus valor litteræ maximæ p invenitur. Verus quidem valor ipſius p tam demum innoteſcit, quando Series in infinitum fuerit continuata; interim tamen ſi jam plures ejus termini fuerint formati, eo propius valor ipſius p cognoſcetur, quo major fuerit terminorum numerus, & quo magis littera iſta p excedat reliquas q, r, s, &c.· perinde vero eſt utrum hæc maxima littera p fuerit ſigno + an ſigno — affecta, quoniam ejus Poteſtates æque increſcunt.

337. Quemadmodum nunc hæc inveſtigatio ad inventionem radicum æquationis cujuſvis algebraïcæ accommodari

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poſſit, ſatis eſt perſpicuum. Ex Factoribus enim denominatoris $1 - \alpha z - \beta z z - \gamma z^3 - \delta z^4 -$ &c., cognitis facile aſſignantur radices æquationis hujus

$$1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c. = 0,$$

ita ut, ſi Factor fuerit $1 - pz$, hujus æquationis radix una futura ſit $z = \frac{1}{p}$. Cum igitur ex Serie recurrente reperiatur maximus numerus p, indidem obtinebitur minima radix æquationis $1 - \alpha z - \beta z^2 - \gamma z^3 -$ &c. $= 0$. Vel, ſi ponatur $z = \frac{1}{x}$ ut prodeat hæc æquatio

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

ejuſdem methodi ope eruitur maxima hujus æquationis radix $x = p$.

338. Si igitur proponatur æquatio hæc

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

quæ omnes radices habeat reales & inter ſe inæquales, harum radicum maxima ſequenti modo reperietur. Formetur ex coëfficientibus hujus æquationis fractio

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

Hincque formetur Series recurrens, aſſumendo pro arbitrio numeratorem, ſeu, quod eodem redit, aſſumendo pro libitu terminos initiales; quæ ſit

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots\ldots + Pz^n + Qz^{n+1}$$

dabitque fractio $\frac{Q}{P}$ valorem radicis maximæ x pro æquatione propoſita, eo propius, quo major fuerit numerus n.

Exemplum I.

Sit propoſita iſta æquatio xx — 3x — 1 = o, *cujus maximam radicem inveniri oporteat.*

Formetur fractio $\frac{a + bz}{1 - 3z - zz}$, unde poſitis duobus primis terminis 1, 2, orietur iſta Series recurrens

1, 2, 7, 23, 76, 251, 829, 2738, &c.,

erit ergo $\frac{2738}{829}$ proxime æqualis radici æquationis propoſitæ maximæ. Valor autem hujus fractionis in partibus decimalibus expreſſus eſt

3,' 3027744

æquationis vero radix maxima eſt $= \frac{3 + \sqrt{13}}{2} =$

3, 3027756,

quæ inventam ſuperat tantum una parte millioneſima. Ceterum notandum eſt fractiones $\frac{Q}{P}$ alternatim vera radice eſſe majores & minores.

Exemplum II.

Propoſita ſit iſta æquatio $3x - 4x^3 = \frac{1}{2}$ *cujus radices exhibent Sinus trium Arcuum, quorum triplorum Sinus eſt* $= \frac{1}{2}$.

Æquatione perducta ad hanc formam $o = 1 - 6x \star + 8x^3$, quæratur hujus, ut in numeris integris maneamus, radix minima, ita ut non opus ſit pro x ponere $\frac{1}{2}$. Formetur ergo hæc fractio

$$\frac{a + bx + cxx}{1 - 6x \star + 8x^3}$$

ex qua ſumendis pro lubitu tribus terminis initialibus o, o, 1, quia

quia hoc modo calculus facilime expeditur, orietur hæc Series recurrens, omittendis poteſtatibus ipſius x quia tantum coëfficientibus opus eſt, CAP. XVII.

$$0; 0; 1; 6; 36; 208; 1200; 6912; 39808; 229248.$$

Erit ergo proxime æquationis radix minima $= \frac{39808}{229248} = \frac{311}{1791} = 0,1736515$, quæ propterea eſſe deberet Sinus anguli 10°; hic autem ex tabulis eſt $0,1736482$, quem ſuperat radix inventa parte $\frac{33}{100000000}$. Facilius autem hæc eadem radix inveniri poteſt ponendo $x = \frac{1}{2}y$, ut prodeat æquatio $1 - 3y * + y^3 = 0$, ex qua ſimili modo tractata oritur Series

$$0, 0, 1, 3, 9, 26, 75, 216, 622, 1791, 5157 \text{ \&c.},$$

erit ergo proxime æquationis radix minima $y = \frac{1791}{5157} = \frac{199}{573} = 0,3472949$, unde fit $x = \frac{1}{2}y = 0,1736479$, qui valor decies propius accedit quam præcedens.

EXEMPLUM III.

Si deſideretur ejuſdem æquationis propoſitæ $0 = 1 - 6x * + 8x^3$, *radix maxima.*

Ponatur $x = \frac{y}{2}$, eritque $y^3 * - 3y + 1 = 0$. Cujus æquationis radix maxima reperietur per Seriem recurrentem cujus ſcala relationis eſt $0, 3 - 1$, unde ergo oritur, ſumtis tribus terminis initialibus pro arbitrio,

$$1, 1, 1, 2, 2, 5, 4, 13, 7, 35, 8, 98 - 11, \text{ \&c.},$$

in qua Serie cum ad terminos negativos perveniatur, id indicio eſt maximam radicem eſſe negativam, eſt enim $x = -$

LIB. I. *ſin.* $70^0 = - 0,9396926$. Quare hujus ratio in terminis initialibus eſt habenda, hoc modo

$$1-2+4-7+14-25+49-89+172-316+605-\&c.,$$

ex qua erit $y = \frac{-605}{316}$ & $x = \frac{-605}{632} = -0,957$, quæ a veritate vehementer abludit.

339. Ratio hujus diſſenſus potiſſimum eſt, quod æquationis propoſitæ radices ſint *ſin.* 10^0, *ſin.* 50^0, & $-$*ſin.* 70^0, quarum binæ maximæ tam parum a ſe invicem diſcrepant, ut in Poteſtatibus, ad quas Seriem continuavimus, ſecunda radix *ſin.* 50^0 adhuc notabilem teneat rationem ad radicem maximam, ideoque præ ea non evaneſcant. Hincque etiam ſaltus pendet, quod alternatim valores inventi fiant nimis magni & nimis parvi: Sic, ſumendo

$$y = \frac{-316}{172}, \text{ fit } x = \frac{-158}{172} = \frac{-79}{86} = -0,918.$$

Nam, quoniam Poteſtates radicis maximæ alternatim fiunt affirmativæ & negativæ, alternatim quoque Poteſtates ſecundæ radicis adduntur & tolluntur: quamobrem, quo hæc diſcrepantia fiat inſenſibilis, Series vehementer ulterius debet continuari.

340. Aliud vero remedium huic incommodo afferri poteſt, tranſmutando æquationem ope idoneæ ſubſtitutionis in aliam formam, cujus radices ſibi non amplius ſint tam vicinæ. Sic, ſi in æquatione $0 = 1 - 6x + 8x^3$ cujus radices ſunt $-$*ſin.* 70^0, $+$*ſin.* 50^0, $+$*ſin.* 10^0, ponatur $x = y - 1$, æquationis $0 = 8y^3 - 24yy + 18y - 1$ radices erunt $1 -$*ſin.* 70^0, $1 +$*ſin.* 50^0; $1 +$*ſin.* 10^0.; ideoque ejus radix minima erit $1 -$*ſin.* 70^0, cum tamen hæc *ſin.* 70^0 eſſet radix maxima æquationis præcedentis; atque $1 +$*ſin.* 50^0 nunc eſt radix maxima, cum *ſin.* 50^0 ante eſſet media. Atque hoc modo quævis radix per ſubſtitutionem in maximam minimamve radicem novæ æqua-

tionis tranſmutari, ideoque per methodum hic traditam inveniri poterit. Quia præterea in hoc exemplo radix $1 - \mathit{ſin}.\, 70^\circ$ multo minor eſt, quam binæ reliquæ, etiam facile per Seriem recurrentem proxime cognoſcetur. CAP. XVII.

EXEMPLUM IV.

Invenire radicem minimam æquationis $0 = 8y^3 - 24yy + 18y - 1$, *quæ ab unitate ſubtracta relinquet Sinum anguli* 70°.

Ponatur $y = \frac{1}{2}z$, ut ſit $0 = z^3 - 6zz + 9z - 1$, cujus radix minima invenietur per Seriem recurrentem, cujus ſcala relationis eſt $9, -6, +1$, pro radice autem maxima invenienda ſcala relationis ſumi deberet $6, -9, +1$. Pro minima ergo formetur hæc Series

$1, 1, 1, 4, 31, 256, 2122, 17593; 145861;$ &c.,

erit ergo proxime $z = \frac{17593}{145861} = 0,12061483$ & $y = 0,06030741$, atque $\mathit{ſin}.\, 70^\circ = 1 - y = 0,93969258$, quæ a veritate ne in ultima quidem figura diſcrepat. Ex hoc ergo exemplo intelligitur quantam utilitatem idonea transformatio æquationis ope ſubſtitutionis ad inventionem radicum afferat, & quod hoc pacto methodus tradita non ſolum ad maximas minimaſve radices adſtringatur, ſed etiam omnes radices exhibere queat.

341. Cognita ergo jam quacunque æquationis propoſitæ radice proxime, ita ut, verbi gratia, numerus k quam minime a quapiam radice differat, ponatur $x - k = y$ ſeu $x = y + k$, hocque modo prodibit æquatio, cujus radix minima erit $= x - k$, quæ igitur ſi per Series recurrentes indagetur, quod facillime fiet, quia hæc radix multo minor erit, quam ceteræ, ſi ea ad k addatur habebitur radix vera ipſius x, pro æquatione propoſita. Hoc vero artificium tam late patet, ut etiamſi æquatio contineat radices imaginarias, uſum ſuum retineat.

342. Imprimis autem ſine hoc artificio radix cognoſci nequit, cui datur alia æqualis ſed ſigno contrario affecta. Scilicet, ſi æquatio cujus maxima radix p, eamdem radicem habeat $-p$, tum, etiamſi Series recurrens in infinitum continuetur, tamen radix hæc p nunquam obtinebitur. Sit, ut hoc exemplo illuſtremus, propoſita æquatio $x^3 - x^2 - 5x + 5 = 0$, cujus maxima radix eſt $\sqrt{5}$, præter quam vero ineſt quoque $-\sqrt{5}$. Si igitur modo ante præſcriptio, pro radice maxima invenienda, utamur, atque Seriem recurrentem formemus ex ſcala relationis $1, +5, -5$, quæ erit

$$1, 2, 3, 8, 13, 38, 63, 188, 313, 938, 1563, \&c.,$$

ubi ad nullam rationem conſtantem pervenitur. Termini vero alterni rationem æquabilem induunt, quorum ſi quiſque per præcedentem dividatur, reperietur quadratum maximæ radicis, ſic enim eſt proxime $5 = \frac{1563}{313} = \frac{938}{188} = \frac{313}{63}$. Quoties ergo termini tantum alterni ſeſe ad rationem conſtantem componunt, toties quadratum radicis quæſitæ proxime obtinetur. Ipſa autem radix $x = \sqrt{5}$ invenitur ponendo $x = y + 2$ unde fit $1 - 3y - 5yy - y^3 = 0$, cujus radix minima cognoſcetur ex Serie

$$1, 1, 1, 9, 33, 145, 609, 2585, 10945, \&c.,$$

erit enim proxime $= \frac{2585}{10945} = 0,2361$, at $2,2361$ eſt proxime $= \sqrt{5}$, quæ eſt radix maxima æquationis.

343. Quanquam numerator fractionis, ex qua Series recurrens formatur, a noſtro arbitrio pendet, tamen idonea ejus conſtitutio plurimum confert, ut valor radicis cito vero proxime exhibeatur. Cum enim aſſumtis, ut ſupra, Factoribus denominatoris (334), ſit terminus generalis Seriei recurrentis $= z^n(Ap^n + Bq^n + Cr^n + \&c.)$, iſti coëfficientes A, B, C, &c., per numeratorem fractionis determinantur; unde fieri poteſt, ut A ſive magnum ſive parvum valorem obtineat: priori caſu radix maxima p cito reperitur, poſteriore vero tarde.

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Quin etiam numerator ita accipi poteſt ut A prorſus evaneſcat, quo caſu, etiamſi Series in infinitum continuetur, tamen nunquam radicem maximam p præbebit. Hoc autem evenit ſi numerator ita accipiatur, ut ipſe eundem habeat Factorem $1 - pz$, ſic enim ex computo penitus tolletur. Sic, ſi proponatur æquatio $x^3 - 6xx + 10x - 3 = 0$, cujus maxima radix eſt $= 3$, indeque formetur fractio CAP. XVII.

$$\frac{1 - 3z}{1 - 6z + 10z^2 - 3z^3}$$

ut Seriei recurrentis ſit ſcala relationis $6, -10, +3$

$$1 + 3, 8, 21, 55, 144, 377, \&c.,$$

cujus termini prorſus non convergunt ad rationem, $1 : 3$. Eadem enim Series oritur ex fractione $\frac{1}{1 - 3z + zz}$, ac propterea maximam radicem æquationis $x^2 - 3x + 1 = 0$ exhibet.

344. Quin etiam numerator ita aſſumi poteſt, ut per Seriem recurrentem quævis radix æquationis reperiatur, quod fiet ſi numerator fuerit productum ex omnibus Factoribus denominatoris præter eum, cui reſpondet radix quam velimus. Sic, ſi in priori exemplo ſumatur numerator $1 - 3z + zz$, fractio $\frac{1 - 3z + zz}{1 - 6z + 10z^2 - 3z^3}$, dabit hanc Seriem recurrentem $1, 3, 9, 27, 81, 243$, &c., quæ, cum ſit geometrica, ſtatim monſtrat radicem $x = 3$. Fractio enim illa æqualis eſt huic ſimplici $\frac{1}{1 - 3z}$. Hinc apparet, ſi termini initiales, quos pro lubitu aſſumere licet, ita accipiantur, ut progreſſionem geometricam conſtituant, cujus Exponens æquetur uni radici æquationis, tum totam Seriem recurrentem fore geometricam, ideoque eam ipſam radicem eſſe exhibituram, etiamſi neque ſit maxima neque minima.

345. Ne igitur, dum quærimus radicem vel maximam vel minimam, præter expectationem nobis alia radix per Seriem recurrentem exhibeatur, hujuſmodi numerator debet eligi, qui

 cum denominatore nullum Factorem habeat communem, quod fiet ſi pro numeratore unitas accipiatur, unde terminus primus Seriei erit $= 1$, ex quo ſolo ſecundum ſcalam relationis ſequentes omnes definiantur. Hocque modo ſemper certe radix æquationis vel maxima vel minima, prout fuerit propoſitum, eruetur. Sic, propoſita æquatione $y^3 * - 3y + 1 = 0$, cujus radix maxima deſideratur, ex ſcala relationis $0, +3, -1$ incipiendo ab unitate ſequens oritur Series recurrens

$$1 - 0 + 3 - 1 + 9 - 6 + 28 - 27 + 90 - 109 + 297, \\ - 517 + 1000 - 1848 + 3517 - 6544 + \&c.,$$

quæ manifeſto ad rationem conſtantem convergit, oſtenditque radicem maximam eſſe negativam, atque proxime $y = \frac{-6544}{3517}$ $= -1,860676$, quæ eſſe debebat $= -1,8679385 2$. Ratio autem ſupra eſt allata, cur tam lente ad verum valorem appropinquetur, propterea quod altera radix non multo ſit minor maxima, ſimulque ſit affirmativa.

346. His probe perpenſis, quæ cum in genere tum ad exempla allata monuimus, ſumma utilitas hujus methodi ad inveſtigandas æquationum radices luculenter perſpicietur; artificia vero, quibus operatio contrahi, eoque promtior reddi queat, ſatis quoque ſunt indicata; ita ut nihil inſuper addendum eſſet, niſi caſus, quibus æquatio vel radices habet æquales vel imaginarias, evolvendi ſupereſſent. Ponamus ergo denominatorem fractionis

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

habere Factorem $(1 - pz)^2$, reliquis Factoribus exiſtentibus $1 - qz$, $1 - rz$, &c.. Seriei ergo recurrentis hinc natæ terminus generalis erit $= z^n ((n+1) Ap^n + Bp^n + Cq^n +$ &c.), quæ cujuſmodi valorem ſit adeptura, ſi n fuerit numerus

vehementer magnus, duo caſus ſunt diſtinguendi, alter quo p eſt numerus major reliquis q, r, &c., alter quo p non præbet radicem maximam. Caſu priori, quo p ſimul eſt radix maxima, ob coëfficientem $(n+1)$ reliqui termini $Bp^n + Cq^n$ &c., non tam cito præ eo evaneſcent, quam ante: ſin autem q fuerit $> p$, tum quoque tarde terminus $(n+1)\,Ap^n$ præ Bq^n evaneſcet, ideoque inveſtigatio radicis maximæ admodum evadet moleſta. CAP. XVII.

EXEMPLUM I.

Sit propoſita æquatio $x - 3xx + 4 = 0$, *cujus maxima radix* 2 *bis occurrit.*

Quæratur ergo maxima radix hæc modo ante expoſito per evolutionem fractionis

$$\frac{1}{1 - 3z * + 4z^3}$$

quæ dabit hanc Seriem recurrentem

1, 3, 9, 23, 57, 135, 313, 711, 1593, &c.;

ubi quidem quivis terminus per præcedentem diviſus dat quotum binario majorem. Cujus ratio ex termino generali facillime patet, rejectis enim in eo terminis Bp^n, Cq^n &c., erit terminus poteſtati z^n reſpondens $= (n+1)\,Ap^n + Bp^n$, ſequens $= (n+2)\,Ap^{n+1} + Bp^{n+1}$, qui per illum diviſus dat $\frac{(n+2)\,A + B}{(n+1)\,A + B}\,p > p$, niſi n jam in infinitum excreverit.

EXEMPLUM II.

Sit jam propoſita æquatio $x^3 - xx - 5x - 3 = 0$, *cujus maxima radix* $= 3$, *reliquæ duæ æquales* $= -1$, & quæratur maxima radix ope Seriei recurrentis, cujus ſcala relationis eſt $1, +5, +3$; unde oritur

$$1, 1, 6, 14, 47, 135, 412, 1228, \&c.,$$

quæ ideo ſatis cito valorem 3 exhibet, quod Poteſtates minoris radicis -1, etiamſi multiplicentur per $n + 1$, tamen mox præ Poteſtatibus ipſius 3 evaneſcant.

EXEMPLUM III.

Sin antem proponeretur æquatio $x^3 + xx - 8x - 12 = 0$, cujus radices ſunt $3, -2, -2$, multo tardius maxima ſeſe prodet. Orietur enim hæc Series

$$1, -1, 9, -5, 65, 3, 457, 347, 3345, 4915, \&c.;$$

quæ adhuc longiſſime continuari deberet, antequam pateret, radicem inde oriundam eſſe $= 3$.

347. Simili modo ſi tres Factores eſſent æquales, ita ut denominatoris Factor unus eſſet $(1 - pz)^3$, reliqui $1 - qz$, $1 - rz$, &c., Seriei recurrentis terminus generalis erit $= z^n \left(\frac{(n+1)(n+2)}{1 \cdot 2} Ap^n + (n+1) Bp^n + Cp^n + Dq^n + Cr^n \&c.\right)$ Si ergo p fuerit maxima radix, atque n fuerit numerus tantus, ut Poteſtates q^n, r^n &c. præ p^n evaneſcant, tum ex Serie recurrente orietur radix $=$

$$\frac{\frac{1}{2}(n+2)(n+3)A + (n+2)B + C}{\frac{1}{2}(n+1)(n+2)A + (n+1)B + C} p,$$

quæ, niſi ſit n numerus maximus & quaſi infinitus, verum ipſius

ipſius p valorem indicabit. Erit autem iſte radice valor $= p + \frac{(n+2)A+B}{\frac{1}{2}(n+1)(n+2)A+(n+1)B+C}p.$ CAP. XVII.

Quod ſi autem p non fuerit radix maxima, tum inventio maxime multo magis adhuc impedietur; unde ſequitur æquationes, quæ contineant radices æquales, hac methodo per Series recurrentes multo difficilius reſolvi, quam ſi omnes radices eſſent inter ſe inæquales.

348. Videamus nunc quomodo Series recurrens in infinitum continuata debeat eſſe comparata, quando denominator fractionis habet Factores imaginarios. Sint igitur fractionis

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

Factores denominatoris reales $1 - qz$, $1 - rz$, &c., inſuperque Factor trinomialis $1 - 2pz.\ cof.\ \varphi + ppzz$ continens duos Factores ſimplices imaginarios. Quod ſi ergo Series recurrens ex illa fractione orta fuerit

$A + Bz + Cz^2 + Dz^3 + \ldots\ldots\ldots\ldots + Pz^n + Qz^{n+1}$;
erit, per ea quæ ſupra expoſuimus, coëfficiens $P =$
$\frac{A.\ fin.\ (n+1)\varphi + B.\ fin.\ n\varphi}{fin.\ \varphi}\ p^n + Cq^n + Dr^n + \&c.$. Si igitur numerus p minor fuerit, quam unus ceterorum q, r, &c., ita ut maxima radix æquationis

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \&c. = 0;$$

ſit realis, tum ea per Series recurrentes æque reperietur, ac ſi nullæ radices ineſſent imaginariæ.

349. Inventio ergo maximæ radicis realis per radices imaginarias non perturbabitur, ſi hæ ita fuerint comparatæ, ut binarum, quæ Factorem realem componunt, productum non ſit

LIB. I. majus quadrato radicis maximæ. Sin autem binæ ejuſmodi inſint radices imaginariæ, ut earum productum adæquet vel adeo ſuperet quadratum maximæ radicis realis, tum inveſtigatio ante expoſita nihil declarabit, propterea quod Poteſtas p^n, præ ſimili Poteſtate radicis maximæ nunquam evaneſcit, etiamſi Series in infinitum continuetur. Cujus exempla illuſtrationis cauſa hic adjicere viſum eſt.

EXEMPLUM I.

Sit propoſita æquatio $x^3 - 2x - 4 = 0$, *cujus radicem maximam inveſtigari oporteat.*

Reſolvitur hæc æquatio in duos Factores $(x - 2)(xx + 2x + 2)$; unde unam habet radicem realem 2 & duas reliquas imaginarias, quarum productum eſt 2, minus quam quadratum radicis realis. Quam ob rem ea per modum hactenus traditum cognoſci poterit. Formetur ergo Series recurrens ex ſcala relationis 0, + 2, + 4, quæ erit

1, 0, 2, 4, 4, 16, 24, 48, 112, 192, 416, 832, &c.;

unde ſatis luculenter radix realis 2 cognoſci poteſt.

EXEMPLUM II.

Propoſita ſit æquatio $x^3 - 4xx + 8x - 8 = 0$, *cujus radix una realis eſt* 2, *binarum imaginariarum productum vero* $= 4$, *ideòque æquale quadrato radicis realis* 2.

Quæramus ergo radicem per Seriem recurrentem, quod quo facilius fieri queat, ponamus $x = 2y$, ut habeatur $y^3 - 2yy + 2y - 1 = 0$, unde formetur Series recurrens

1, 2, 2, 1, 0, 0, 1, 2, 2, 1, 0, 0, 1, 2, 2, 1, &c.;

in qua cum iidem termini perpetuo revertantur, nihil inde

aliud colligi poteſt, niſi radicem maximam vel non eſſe realem, vel dari imaginarias, quarum productum æquale ſit aut ſuperet quadratum radicis realis. CAP. XVII.

EXEMPLUM III.

Sit jam propoſita æquatio $x^3 - 3xx + 4x - 2 = 0$, *cujus radix realis eſt* 1, *imaginariarum vero productum* $= 2$.

Formetur ergo ex ſcala relationis 3, -4, $+2$, Series 1, 3, 5, 5, 1, -7, -15, -15; $+1$, 33, 65, 65, 1, &c., in qua cum termini modo fiant affirmativi, modo negativi, radix realis 1 inde nullo modo cognoſci poterit. Hujuſmodi vero revolutiones ſemper oſtendunt radicem, quam Series præbere debebat, eſſe imaginariam; hic enim radices imaginariæ poteſtate ſunt majores quam realis 1.

350. Sit igitur in fractione generali productum binarum radicum imaginariarum pp majus quam ullius radicis realis quadratum, ita ut præ p^n reliquæ poteſtates q^n, r^n, &c., evaneſcant ſi n ſit numerus infinitus. Hoc ergo caſu fiet $P = \frac{A.ſin.(n+1)\varphi + B.ſin.n\varphi}{ſin.\varphi} p^n$, & $Q = \frac{A.ſin.(n+2)\varphi + B.ſin.(n+1)\varphi}{ſin.\varphi} p^{n+1}$ ideoque $\frac{Q}{P} = \frac{A.ſin.(n+2)\varphi + B\,ſin.(n+1)\varphi}{A.ſin.(n+1)\varphi + B.ſin.n\varphi} p$. Quæ expreſſio nunquam valorem conſtantem induet, etiamſi n ſit numerus infinitus. Sinus enim Angulorum perpetuo maxime manent mutabiles, ita ut mox ſint affirmativi mox negativi.

351. Interim tamen ſi fractiones ſequentes $\frac{R}{Q}$, $\frac{S}{R}$ ſimili modo ſumantur, indeque litteræ A & B eliminentur, ſimul numerus n ex calculo egredietur; reperietur enim $Ppp + R = 2Qp.\,coſ.\varphi$, unde fit $coſ.\varphi = \frac{Ppp+R}{2Qp}$; ſimiliter vero erit

LIB. I. $cos. \varphi = \frac{Qpp + S}{2Rp}$, ex quorum duorum valorum comparatione fit $p = \surd \frac{RR - QS}{QQ - PR}$, atque $cos. \varphi = \frac{QR - PS}{2\surd(Q^2 - PR)(R^2 - QS)}$. Quam ob rem ſi Series recurrens jam eo uſque fuerit continuata, ut præ p^n reliquarum radicum Poteſtates evaneſcant, tum hoc modo Factor trinomialis $1 - 2pz. cos. \varphi + ppzz$ poterit inveniri.

352. Quoniam iſte calculus non ſatis exercitatis moleſtiam creare poſſet, eum totum hic apponam. Ex valore ipſius $\frac{Q}{P}$ invento oritur $APp. ſin. (n+2)\varphi + BPp. ſin. (n+1)\varphi = AQ. ſin. (n+1)\varphi + BQ. ſin. n\varphi$, unde fit $\frac{A}{B} = \frac{Q. ſin. n\varphi - Pp. ſin. (n+1)\varphi}{Pp. ſin. (n+2)\varphi - Q. ſin. (n+1)\varphi}$. Pari ratione erit $\frac{A}{B} = \frac{R. ſin. (n+1)\varphi - Qp. ſin. (n+2)\varphi}{Qp. ſin. (n+3)\varphi - R. ſin. (n+2)\varphi}$; æquatis his duobus valoribus fiet

$0 = QQp. ſin. n\varphi. ſin. (n+3)\varphi - QR. ſin. n\varphi. ſin. (n+2)\varphi - PQpp. ſin. (n+1)\varphi. ſin. (n+3)\varphi - QQp. ſin. (n+1)\varphi. ſin. (n+2)\varphi + QR. ſin. (n+1)\varphi. ſin. (n+1)\varphi + PQpp. ſin. (n+1)\varphi. ſin. (n+2)\varphi$.

Cum autem ſit $ſin. a. ſin. b = \frac{1}{2}. cos. (a-b) - \frac{1}{2}. cos. (a+b)$ fiet $0 = \frac{1}{2}QQp. (cos. 3\varphi - cos. \varphi) + \frac{1}{2}QR. (1 - cos. 2\varphi) + \frac{1}{2}PQpp. (1 - cos. 2\varphi)$ quæ per $\frac{1}{2}Q$ diviſa dat $(Ppp + R)(1 - cos. 2\varphi) = Qp. (cos. \varphi - cos. 3\varphi)$. At eſt $cos. \varphi = cos. 2\varphi. cos. \varphi + ſin. 2\varphi. ſin. \varphi$ & $cos. 3\varphi = cos. 2\varphi\, cos. \varphi - ſin. 2\varphi. ſin. \varphi$ unde $cos. \varphi - cos. 3\varphi = 2 ſin. 2\varphi. ſin. \varphi = 4 ſin. \varphi^2 \times cos. \varphi$ & $1 - cos. \varphi = 2 ſin. \varphi^2$, ex quo erit $Ppp + R = 2Qp. cos. \varphi$, & $cos. \varphi = \frac{Ppp + R}{2Qp}$, atque $cos. \varphi = \frac{Qpp + S}{2Rp}$: unde

ſuperiores valores prodeunt, ſcilicet $p = \sqrt{\frac{RR - QS}{QQ - PS}}$ & $coſ.\,\varphi = \frac{QR - PS}{2\sqrt{(Q^3 - PR)(RR - QS)}}$.

353. Si denominator fractionis, ex qua Series recurrens formatur, plures habeat Factores trinomiales inter ſe æquales, tum, ſpectata forma termini generalis ſupra data, patebit inventionem radicum multo magis fieri incertam. Interim tamen ſi una quæcunque radix realis jam proxime fuerit detecta, tum æquationis transformatione ſemper valor ejuſdem radicis multo propior eruetur. Ponatur enim x æqualis valori illi jam detecto $+ y$, atque novæ æquationis quæratur minima radix pro y, quæ addita ad illum valorem præbebit verum ipſius x valorem.

Exemplum.

Sit propoſita iſta æquatio $x^3 - 3xx + 5x - 4 = 0$, *cujus unam radicem fere eſſe* $= 1$ *inde conſtat, quod, poſito* $x = 1$, *prodit* $x^3 - 3xx + 5x - 4 = -1$.

Ponatur ergo $x = 1 + y$, fietque $1 - 2y - y^3 = 0$, unde pro radice minima invenienda formetur Series recurrens, cujus ſcala relationis $2, 0, +1$, quæ erit

$$1, 2, 4, 9, 20, 44, 97, 214, 472, 1041, 2296, \&c.,$$

unde radix minima ipſius y erit proxime $\frac{1041}{2296} = 0,453397$, ita ut ſit $x = 1,453397$, qui valor tam prope vix alia methodo æque facile obtineri poterit.

354. Quod ſi autem Series quæcunque recurrens tandem tam prope ad progreſſionem geometricam convergat, tum ex ipſa lege progreſſionis ſtatim facile cognoſci poterit, cujuſnam æquationis radix ſit futura quotus qui ex diviſione unius termini per præcedentem oritur. Sint

$$P,\ Q,\ R,\ S,\ T,\ \&c.,$$

termini Seriei recurrentis a principio jam longiſſime remoti; ita ut cum progreſſione geometrica confundantur; ſitque $T = \alpha S + \beta R + \gamma Q + \delta P$, ſeu ſcala relationis $\alpha, +\beta, +\gamma, +\delta$. Ponatur valor fractionis $\frac{Q}{P} = x$; erit $\frac{R}{P} = xx$; $\frac{S}{P} = x^3$ & $\frac{T}{P} = x^4$, qui in ſuperiori æquatione ſubſtituti dabunt

$$x^4 = \alpha x^3 + \beta x^2 + \gamma x + \delta,$$

unde patet quotum $\frac{Q}{P}$ tandem præbere radicem unam æquationis inventæ. Hoc vero & præcedens methodus indicat, præterea vero docet fractionem $\frac{Q}{P}$ dare maximam æquationis radicem.

355. Poteſt quoque hæc methodus inveſtigandarum radicum ſæpenumero utiliter adhiberi, ſi æquatio ſit infinita. Ad quod oſtendendum propoſita ſit æquatio $\frac{1}{2} = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \&c.$, cujus radix minima z exhibet Arcum 30°, ſeu Semiperipheriæ Circuli ſextantem. Perducatur ergo æquatio ad hanc formam

$$1 - 2z + \frac{z^3}{3} - \frac{z^5}{60} + \frac{z^7}{2520} - \&c. = 0.$$

Hinc ergo formetur Series recurrens, cujus ſcala relationis eſt infinita, ſcilicet

$$2,\ 0,\ -\frac{1}{3},\ 0,\ +\frac{1}{60},\ 0,\ -\frac{1}{2520},\ 0\ \&c.,$$

eritque Series recurrens

$$1,\ 2,\ 4,\ \frac{23}{3},\ \frac{44}{3},\ \frac{1681}{60},\ \frac{2408}{45},\ \&c.,$$

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erit ergo proxime $z = \frac{1681.45}{2408.60} = \frac{1681.3}{2408.4} = \frac{5043}{9632} = 0,52356$:
At ex proportione Peripheriæ ad Diametrum cognita debebat eſſe $z = 0,523598$, ita ut radix inventa tantum parte $\frac{3}{100000}$ a vero diſcrepet. Hoc autem in hac æquatione commode uſu venit, quod ejus omnes radices ſint reales, atque a minima reliquæ ſatis notabiliter diſcrepent. Quæ conditio cum rariſſime in æquationibus infinitis locum habeat, huic methodo ad eas reſolvendas parum uſus relinquitur.

CAPUT XVIII.

De fractionibus continuis.

356. Quoniam in præcedentibus Capitibus plura, cum de Seriebus infinitis, tum de productis ex infinitis Factoribus conflatis diſſerui, non incongruum fore viſum eſt, ſi etiam nonnulla de tertio quodam expreſſionum infinitarum genere addidero, quod continuis fractionibus vel diviſionibus continetur. Quanquam enim hoc genus parum adhuc eſt excultum, tamen non dubitamus, quin ex eo ampliſſimus uſus in analyſin infinitorum aliquando ſit redundaturus. Exhibui enim jam aliquoties ejuſmodi ſpecimina, quibus hæc expectatio non parum probabilis redditur. Imprimis vero ad ipſam Arithmeticam & Algebram communem non contemnenda ſubſidia affert iſta ſpeculatio, quæ hoc Capite breviter indicare atque exponere conſtitui.

357. Fractionem autem continuam voco ejuſmodi fractionem, cujus denominator conſtat ex numero integro cum fractione, cujus denominator denuo eſt aggregatum ex integro & fractione, quæ porro ſimili modo ſit comparata, ſive iſta affectio in infinitum progrediatur ſive alicubi ſiſtatur. Hujuſmodi ergo fractio continua erit ſequens expreſſio

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$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\&c.}}}}} \quad \text{vel}\; a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\epsilon}{f+\&c.}}}}};$$

in quarum forma priori omnes fractionum numeratores ſunt unitates, quam potiſſimum hic contemplabor, in altera vero forma ſunt numeratores numeri quicunque.

358. Expoſita ergo fractionum harum continuarum forma; primum videndum eſt, quemadmodum earum ſignificatio conſueto more expreſſa inveniri queat. Quæ ut facilius inveniri poſſit, progrediamur per gradus, abrumpendo illas fractiones primo in prima, tum in ſecunda, poſt in tertia & ita porro fractione; quo facto patebit fore

$$a = a$$

$$a+\frac{1}{b} = \frac{ab+1}{b}$$

$$a+\cfrac{1}{b+\cfrac{1}{c}} = \frac{abc+a+c}{bc+1}$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}} = \frac{abcd+ab+ad+cd+1}{bcd+b+d}$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e}}}} = \frac{abcde+abe+ade+cde+abc+a+c+e}{bcde+be+de+bc+1}$$

&c..

359. Etſi in his fractionibus ordinariis non facile lex, ſecundum quam numerator ac denominator ex litteris a, b, c, d, &c., componantur, perſpicitur, tamen attendenti ſtatim patebit, quemadmodum quælibet fractio ex præcedentibus formari queat. Quilibet enim numerator eſt aggregatum ex numeratore ultimo per novam litteram multiplicato, & ex numeratore

numeratore penultimo ſimplici : eademque lex in denominatoribus obſervatur. Scriptis ergo ordine litteris a, b, c, d, &c., ex iis fractiones inventæ facile formabuntur hoc modo

$$\begin{array}{ccccc} a & b & c & d & e \\ \frac{1}{0}; & \frac{a}{1}; & \frac{ab+1}{b}; & \frac{abc+a+c}{bc+1}; & \frac{abcd+ab+ad+cd+1}{bcd+b+d} \end{array}$$

ubi quilibet numerator invenitur, ſi præcedentium ultimus per indicem ſupra ſcriptum multiplicetur atque ad productum antepenultimus addatur; quæ eadem lex pro denominatoribus valet. Quo autem hac lege ab ipſo initio uti liceat, præfixi fractionem $\frac{1}{0}$ quæ, etiamſi e fractione continua non oriatur, tamen progreſſionis legem clariorem efficit. Quælibet autem fractio exhibet valorem fractionis continuæ uſque ad eam litteram, quæ antecedenti imminet, incluſive continuata.

360. Simili modo altera fractionum continuarum forma

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{\&c.}}}}}},$$

dabit, prout aliis aliiſque locis abrumpitur, ſequentes valores

$$a = a$$

$$a+\frac{\alpha}{b} = \frac{ab+\alpha}{b}$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c}} = \frac{abc+\beta a+\alpha c}{bc+\beta}$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d}}} = \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}$$

&c.,

Lib. I. quarum fractionum quæque ex binis præcedentibus ſequentem in modum invenietur

$$\overset{a}{\underset{\alpha}{\frac{1}{0}}};\ \overset{b}{\underset{\beta}{\frac{a}{1}}};\ \overset{c}{\underset{\gamma}{\frac{ab+\alpha}{b}}};\ \overset{d}{\underset{\delta}{\frac{abc+\beta a+\alpha c}{bc+\beta}}};\ \overset{e}{\underset{\epsilon}{\frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}}}.$$

361. Fractionibus ſcilicet formandis ſupra inſcribantur indices a, b, c, d, &c., infra autem ſubſcribantur indices $\alpha, \beta, \gamma, \delta$, &c.. Prima fractio iterum conſtituatur $\frac{1}{0}$, ſecunda $\frac{a}{1}$, tum ſequentium quævis formabitur ſi antecedentium ultimæ numerator per indicem ſupra ſcriptum, penultimæ vero numerator per indicem infra ſcriptum multiplicetur & ambo producta addantur, aggregatum erit numerator fractionis ſequentis: ſimili modo ejus denominator erit aggregatum ex ultimo denominatore per indicem ſupra ſcriptum, & ex penultimo denominatore per indicem infra ſcriptum multiplicatis. Quælibet vero fractio hoc modo inventa præbebit valorem fractionis continuæ ad eum uſque denominatorem, qui fractioni antecedenti eſt inſcriptus, continuatæ incluſive.

362. Quod ſi ergo hæ fractiones eouſque continuentur quoad fractio continua indices ſuppeditet, tum ultima fractio verum dabit valorem fractionis continuæ. Præcedentes fractiones vero continuo propius ad hunc valorem accedent, ideoque perquam idoneam appropinquationem ſuggerent. Ponamus enim verum valorem fractionis continuæ

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\&c.}}}} \quad \text{eſſe} = x;$$

atque manifeſtum eſt fractionem primam $\frac{1}{0}$ eſſe majorem

quam x; ſecunda vero $\frac{a}{1}$ minor erit quam x; tertia $a+\frac{\alpha}{b}$ iterum vero valore erit major; quarta denuo minor, atque ita porro hæ fractiones alternatim erunt majores & minores quam x. Porro autem perſpicuum eſt quamlibet fractionem propius accedere ad verum valorem x quam ulla præcedentium; unde hoc pacto citiſſime & commodiſſime valor ipſius x proxime obtinetur; etiamſi fractio continua in infinitum progrediatur, dummodo numeratores α, β, γ, δ, &c., non nimis creſcant; ſin autem omnes iſti numeratores fuerint unitates, tum appropinquatio nulli incommodo eſt obnoxia.

363. Quo ratio hujus appropinquationis ad verum fractionis continuæ valorem melius percipiatur, conſideremus fractionum inventarum differentias. Ac, prima quidem $\frac{1}{0}$ prætermiſſa, differentia inter ſecundam ac tertiam eſt $=\frac{\alpha}{b}$; quarta a tertia ſubtracta relinquit $\frac{\alpha\beta}{b(bc+\beta)}$; quarta a quinta ſubtracta relinquit $\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma)}$, &c.. Hinc exprimetur valor fractionis continuæ per Seriem terminorum conſuetam hoc modo, ut ſit

$$x=a+\frac{\alpha}{b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}-\&c.,$$

quæ Series toties abrumpitur quoties fractio continua non in infinitum progreditur.

364. Modum ergo invenimus fractionem continuam quamcunque in Seriem terminorum, quorum ſigna alternantur, convertendi, ſi quidem prima littera a evaneſcat. Si enim fuerit

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$$x = \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \cfrac{\varepsilon}{f + \&c.}}}}},$$

erit per ea, quæ modo invenimus,

$$x = \frac{\alpha}{b} - \frac{\alpha\beta}{b(bc+\beta)} + \frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)} - \frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)(bcde+\beta de+\gamma be+\delta bc+\beta\delta)} + \&c.$$

Unde, ſi $\alpha, \beta, \gamma, \delta$, &c. fuerint numeri non creſcentes, uti omnes unitates, denominatores vero a, b, c, d, &c. numeri integri quicunque affirmativi, valor fractionis continuæ exprimetur per Seriem terminorum maxime convergentem.

365. His probe conſideratis, poterit viciſſim Series quæcunque terminorum alternantium in fractionem continuam converti, ſeu fractio continua inveniri cujus valor æqualis ſit ſummæ Seriei propoſitæ. Sit enim propoſita hæc Series

$$x = A - B + C - D + E - F + \&c.,$$

erit, ſingulis terminis cum Serie ex fractione continua orta comparandis.

$$A = \frac{\alpha}{b}; \qquad \text{hincque } \alpha = Ab,$$

$$\frac{B}{A} = \frac{\beta}{bc+\beta}; \qquad \text{unde fit } \beta = \frac{Bbc}{A-B}$$

$$\frac{C}{B} = \frac{\gamma b}{bcd+\beta d+\gamma b}; \qquad \gamma = \frac{Cd(bc+\beta)}{b(B-C)}$$

$$\frac{D}{C} = \frac{\delta(bc+\beta)}{bcde+\beta de+\gamma be+\delta bc+\beta\delta}; \qquad \delta = \frac{De(bcd+\beta d+\gamma b)}{(bc+\beta)(C-D)}$$

&c.

At, cum ſit $\beta = \frac{Bbc}{A-B}$, erit $bc + b = \frac{Abc}{A-B}$; unde

$\gamma = \frac{ACcd}{(A-B)(B-C)}$. Porro ſit $bcd + \mathfrak{C}d + \gamma b = (bc+\mathfrak{C})d+\gamma b = \frac{Abcd}{A-B} + \frac{ACbcd}{(A-B)(B-C)} = \frac{ABbcd}{(A-B)(B-C)}$, unde erit $\frac{bcd+\mathfrak{C}d+\gamma b}{bc+\mathfrak{C}} = \frac{Bd}{B-C}$ & $\delta = \frac{BDde}{(B-C)(C-D)}$; ſimili modo reperietur $\epsilon = \frac{CEef}{(C-D)(D-E)}$ & ita porro.

366. Quo iſta lex clarius appareat, ponamus eſſe

$$P = b$$
$$Q = bc + \mathfrak{C}$$
$$R = bcd + \mathfrak{C}d + \gamma d$$
$$S = bcde + \mathfrak{C}de + \gamma be + \delta bc + \mathfrak{C}\delta$$
$$T = bcdef + \text{\&c.}$$
$$V = bcdefg + \text{\&c.},$$

erit ex lege harum expreſſionum

$$Q = Pc + \mathfrak{C}$$
$$R = Qd + \gamma P$$
$$S = Re + \delta Q$$
$$T = Sf + \epsilon R$$
$$V = Tg + \xi S$$
&c..

Cum igitur his adhibendis litteris ſit

$$x = \frac{\alpha}{P} - \frac{\alpha\mathfrak{C}}{PQ} + \frac{\alpha\mathfrak{C}\gamma}{QR} - \frac{\alpha\mathfrak{C}\gamma\delta}{RS} + \frac{\alpha\mathfrak{C}\gamma\delta\epsilon}{ST} - \text{\&c.}.$$

367. Quoniam ergo ponimus eſſe

$$x = A - B + C - D + E - F + \text{\&c.},$$

erit

$$A = \frac{\alpha}{P};\ \alpha = AP.$$

$$\frac{B}{A} = \frac{\beta}{Q}\,; \quad \beta = \frac{BQ}{A}$$
$$\frac{C}{B} = \frac{\gamma P}{R}\,; \quad \gamma = \frac{CR}{BP}$$
$$\frac{D}{C} = \frac{\delta Q}{S}\,; \quad \delta = \frac{DS}{CQ}$$
$$\frac{E}{D} = \frac{\varepsilon R}{T}\,; \quad \varepsilon = \frac{ET}{DR}$$
&c. &c.

Porro vero differentiis ſumendis habebitur

$$A - B = \frac{\alpha(Q - \beta)}{PQ} = \frac{\alpha c}{Q} = \frac{APc}{Q}$$
$$B - C = \frac{\alpha\beta(R - \gamma P)}{PQR} = \frac{\alpha\beta d}{PR} = \frac{BQd}{R}$$
$$C - D = \frac{\alpha\beta\gamma(S - \delta Q)}{QRS} = \frac{\alpha\beta\gamma e}{QS} = \frac{CRe}{S}$$
$$C - D = \frac{\alpha\beta\gamma\delta(T - \varepsilon R)}{RST} = \frac{\alpha\beta\gamma\delta f}{RT} = \frac{DSf}{T},$$
&c. &c. &c.

Si bini igitur in ſe invicem ducantur, fiet

$$(A-B)(B-C) = ABcd.\frac{P}{R}\,; \;\&\; \frac{R}{P} = \frac{ABcd}{(A-B)(B-C)}$$
$$(B-C)(C-D) = BCde.\frac{Q}{S}\,; \;\&\; \frac{S}{Q} = \frac{BCed}{(B-C)(C-D)}$$
$$(D-E)(D-E) = CDef.\frac{R}{T}\,; \;\&\; \frac{T}{R} = \frac{CDef}{(C-D)(D-E)}$$
&c..

Unde, cum ſit $P = b\,;\; Q = \frac{\alpha c}{A - B} = \frac{Abc}{A - B}$, erit

$$\alpha = Ab$$
$$\beta = \frac{Bbc}{A - B}$$
$$\gamma = \frac{ACcd}{(A - B)(B - C)}$$

$$\delta = \frac{BDde}{(B-C)(C-D)}$$
$$\epsilon = \frac{CEef}{(C-D)(D-E)}$$
&c.

368. Inventis ergo valoribus numeratorum $\alpha, \beta, \gamma, \delta$, &c.; denominatores b, c, d, e, &c., arbitrio noſtro relinquuntur: ita autem eos aſſumi convenit, ut, cum ipſi ſint numeri integri, tum valores integros pro $\alpha, \beta, \gamma, \delta$, &c., exhibeant. Hoc vero pendet quoque a natura numerorum A, B, C, &c., utrum ſint integri an fracti. Ponamus eſſe numeros integros, atque quæſito ſatisfiet ſtatuendo

$$\begin{array}{lcl} b = 1 & & \alpha = A \\ c = A - C & & \beta = B \\ d = B - C & \text{unde fit} & \gamma = AC \\ e = C - D & & \delta = BD \\ f = D - E & & \epsilon = CE \\ \&c. & & \&c. \end{array}$$

Quocirca, ſi fuerit,

$$x = A - B + C - D + E - F + \&c.,$$

idem ipſius x valor per fractionem continuam ita exprimi poterit, ut ſit

$$x = \cfrac{A}{1 + \cfrac{B}{A - B + \cfrac{AC}{B - C + \cfrac{BD}{C - D + \cfrac{CE}{D - E + \&c.}}}}}$$

369. Sin autem omnes termini Seriei ſint numeri fracti, ita ut fuerit

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \&c.,$$

habebuntur pro $\alpha, \beta, \gamma, \delta$, &c., ſequentes valores

 $\alpha = \frac{b}{A}$; $\beta = \frac{Abc}{B - A}$; $\gamma = \frac{B^2 cd}{(B - A)(C - B)}$;
$\delta = \frac{C^2 de}{(C - B)(D - C)}$; $\epsilon = \frac{D^2 ef}{(D - C)(E - D)}$, &c..

Ponatur ergo ut ſequitur

$$\begin{array}{ll} b = A; & \alpha = 1 \\ c = B - A; & \beta = AA \\ d = C - B; & \gamma = BB \\ e = D - C; & \delta = CC \end{array}$$

&c.,

eritque per fractionem continuam

$$x = \frac{1}{A + \frac{AA}{B - A + \frac{BB}{C - B + \frac{CC}{D - C + \&c.}}}}.$$

Exemplum I.

Transformetur hæc Series infinita

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.,$$

in fractionem continuam.

Erit ergo $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c., atque, cum Seriei propoſitæ valor ſit $= l2$, erit

$$l2 = \frac{1}{1 + \frac{1}{1 + \frac{4}{1 + \frac{9}{1 + \frac{16}{1 + \frac{25}{1 + \&c.}}}}}}.$$

Exemplum II.

Transformetur hæc Series infinita

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.,$$

ubi π denotat peripheriam circuli, cujus diameter $= 1$, in fractionem continuam.

Subſtitutis loco A, B, C, D, &c., numeris 1, 3, 5, 7, &c., orietur

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.}}}}};$$

hincque, invertendo fractionem, erit

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.}}}},$$

quæ eſt expreſſio, quam BROUNCKERUS primum pro quadratura circuli protulit.

EXEMPLUM III.

Sit propoſita iſta Series infinita

$$x = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \&c.,$$

quæ, ob $A = m$; $B = m + n$; $C = m + 2n$; &c., in hanc fractionem continuam mutatur

$$x = \cfrac{1}{m + \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \&c.}}}}}$$

ex qua fit, invertendo,

$$\frac{1}{x} - m = \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \&c.}}}}.$$

EXEMPLUM IV.

Quoniam, ſupra §. 178, invenimus eſſe

$$\frac{\pi\ coſ.\ \frac{m\pi}{n}}{n\ ſin.\ \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \&c.,$$

erit, pro fractione continuanda, $A = m$; $B = n - m$; $C = n + m$; $D = 2n - m$; &c., unde fiet

$$\frac{\pi\ coſ.\ \frac{m\pi}{n}}{n\ ſin.\ \frac{m\pi}{n}} = \cfrac{1}{m + \cfrac{mm}{n - 2m + \cfrac{(n-m)^2}{2m + \cfrac{(n+m)^2}{n - 2m + \cfrac{(2n-m)^2}{2m + \cfrac{(2n+m)^2}{n - 2m + \&c.}}}}}}$$

370. Si Series propoſita per continuos Factores progrediatur, ut ſit

$$x = \frac{1}{A} - \frac{1}{AB} + \frac{1}{ABC} - \frac{1}{ABCD} + \frac{1}{ABCDE} - \&c.,$$

tum prodibunt ſequentes determinationes

$$\alpha = \frac{b}{A};\ \beta = \frac{bc}{B-1};\ \gamma = \frac{Bcd}{(B-1)(C-1)};$$
$$\delta = \frac{Cde}{(C-1)(D-1)};\ \varepsilon = \frac{Def}{(D-1)(E-1)};\ \&c.,$$

fiat ergo ut ſequitur,

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$$b = A; \quad \alpha = 1$$
$$c = B - 1; \quad \beta = A$$
$$d = C - 1; \quad \gamma = B$$
$$e = D - 1; \quad \delta = C$$
$$f = E - 1; \quad \epsilon = D$$
&c.,

unde conſequenter fiet

$$x = \cfrac{1}{A + \cfrac{A}{B - 1 + \cfrac{B}{C - 1 + \cfrac{C}{D - 1 + \cfrac{D}{E - 1 + \&c.}}}}}$$

Exemplum I.

Quoniam, poſito e numero cujus Logarithmus eſt $= 1$, ſupra invenimus eſſe

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \&c.,$$

ſeu

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \&c.,$$

hæc ſeries in fractionem continuam convertetur ponendo $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c.: quó ergo facto habebitur

$$1 - \frac{1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \&c.}}}}}};$$

unde, aſymmetria initio rejecta, erit

$$\frac{1}{e - 1} = \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \&c.}}}}}$$

EXEMPLUM II.

Invenimus quoque arcus, qui radio æqualis ſumitur, coſinum eſſe $= 1 - \frac{1}{2} + \frac{1}{2.12} - \frac{1}{2.12.30} + \frac{1}{2.12.30.56} -$ &c.. Si ergo fiat $A = 1$, $B = 2$, $C = 12$, $D = 30$, $E = 56$, &c., atque Coſinus arcus qui radio æquatur, ponatur $= x$; erit

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \text{\&c.}}}}}},$$

ſeu

$$\frac{1}{x} - 1 = \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \text{\&c.}}}}}$$

371. Sit Series inſuper cum geometrica conjuncta, ſcilicet

$$x = A - Bz + Cz^2 - Dz^3 + Ez^4 - Fz^5 + \text{\&c.},$$

erit

$$\alpha = Ab\,;\ \beta = \frac{Abcz}{A - Bz}\,;\ \gamma = \frac{ACcdz}{(A - Bz)(B - Cz)}\,;$$
$$\delta = \frac{BDdez}{(B - Cz)(C - Dz)}\,;\ \varepsilon = \frac{CEefz}{(C - Dz)(D - Ez)}\,;\ \text{\&c.}\,;$$

Ponatur nunc

$$\begin{aligned} b &= 1\,; & \alpha &= A\,. \\ c &= A - Bz\,; & \beta &= Bz \\ d &= B - Cz\,; & \gamma &= ACz \\ e &= C - Dz\,; & \delta &= BDz, \end{aligned}$$

unde fiet

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$$x = \cfrac{A}{1 + \cfrac{Bz}{A - Bz + \cfrac{ACz}{B - Cz + \cfrac{BDz}{C - Dz + \&c.}}}}$$

872. Quo autem hoc negotium generalius abſolvamus, ponamus eſſe

$$x = \frac{A}{L} - \frac{By}{Mz} + \frac{Cy^2}{Nz^2} + \frac{Dy^3}{Oz^3} - \frac{Ey^4}{Pz^4} + \&c.,$$

fietque, comparatione inſtituta,

$$\alpha = \frac{Ab}{L}; \; \beta = \frac{BLbcy}{AMz - BLy}; \; \gamma = \frac{ACM^2cdyz}{(AMz - BLy)(BNz - CMy)};$$
$$\delta = \frac{BDN^2deyz}{(BNz - CMy)(COz - DNy)}; \; \&c.,$$

ſtatuantur valores b, c, d, &c., ſequenti modo

$b = L;$	erit	$\alpha = A$
$c = AMz - BLy;$		$\beta = BLLy$
$d = BNz - CMy;$		$\gamma = ACM^2yz$
$e = COz - DNy;$		$\delta = BDN^2yz$
$f = DPz - EOy;$		$\varepsilon = CEO^2yz$
&c.		&c.

unde Series propoſita per ſequentem fractionem continuam exprimetur

$$x = \cfrac{A}{L + \cfrac{BLLy}{AMz - BLy + \cfrac{ACMMyz}{BNz - CMy + \cfrac{BDNNyz}{COz - DNy + \&c.}}}}$$

373. Habeat denique Series propoſita hujuſmodi formam

$$x = \frac{A}{L} - \frac{ABy}{LMz} + \frac{ABCy^2}{LMNz^2} - \frac{ABCDy^3}{LMNOz^3} + \&c.,$$

atque ſequentes valores prodibunt

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$$\alpha = \frac{Ab}{L}\,;\quad \beta = \frac{Bbcy}{Mz - By}\,;\quad \gamma = \frac{CMcdyz}{(Mz - By)(Nz - Cy)}\,;$$

$$\delta = \frac{DNdeyz}{(Nz - Cy)(Oz - Dy)}\,;\quad \epsilon = \frac{EOefyz}{(Oz - Dy)(Pz - Ey)}\,;$$

&c.,

ad valores ergo integros inveniendos fiat

$b = Lz$;	erit	$\alpha = Az$
$c = Mz - By$;		$\beta = BLyz$
$d = Nz - Cy$;		$\gamma = CMyz$
$e = Oz - Dy$;		$\delta = DNyz$
$f = Pz - Ey$;		$\epsilon = EOyz$
&c.		&c.

Unde valor Seriei propoſitæ ita exprimetur, ut ſit

$$x = \cfrac{Az}{Lz + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \&c.}}}}.$$

Vel, ut lex progreſſionis ſtatim a principio fiat manifeſta, erit

$$\frac{Az}{x} - Ay = Lz - Ay + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \&c.}}}$$

374. Hoc modo innumerabiles inveniri poterunt fractiones continuæ in infinitum progredientes, quarum valor verus exhiberi queat. Cum enim, ex ſupra traditis, infinitæ Series, quarum ſummæ conſtent, ad hoc negotium accommodari queant, unaquæque transformari poterit in fractionem continuam, cujus adeo valor ſummæ illius Seriei eſt æqualis. Exempla, quæ jam hic ſunt allata, ſufficiunt ad hunc uſum oſtendendum: verumtamen optandum eſſet, ut methodus detegeretur, cujus beneficio, ſi propoſita fuerit fractio continua quæcunque, ejus valor immediate inveniri poſſet. Quan-

quam enim fractio continua tranſmutari poteſt in Seriem infinitam, cujus ſumma per methodos cognitas inveſtigari queat, tamen plerumque iſtæ Series tantopere fiunt intricatæ, ut earum ſumma, etiamſi ſit ſatis ſimplex, vix ac ne vix quidem obtineri poſſit. CAP. XVIII.

375. Quo autem clarius perſpiciatur, dari ejuſmodi fractiones continuas, quarum valor aliunde facile aſſignari queat, etiamſi ex Seriebus infinitis, in quas convertuntur, nihil admodum colligere liceat, conſideremus hanc fractionem continuam

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \&c.}}}},$$

cujus omnes denominatores ſunt inter ſe æquales; ſi enim hinc modo ſupra expoſito, fractiones formemus

$$\begin{matrix} 0, & 2, & 2, & 2, & 2, & 2, & 2 \\ \frac{1}{0}, & \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70}, & \&c.: \end{matrix}$$

Hinc autem porro oritur hæc Series

$$x = 0 + \frac{1}{2} - \frac{1}{2.5} + \frac{1}{5.12} - \frac{1}{12.29} + \frac{1}{29.70} - \&c.,$$

vel, ſi bini termini conjungantur, erit

$$x = \frac{2}{1.5} + \frac{2}{5.29} + \frac{2}{29.169} + \&c.,$$

vel

$$x = \frac{1}{2} - \frac{2}{2.12} - \frac{2}{12.70} - \&c.$$

Quin etiam, cum ſit

$$x = \frac{1}{4} - \frac{1}{2.2.5} + \frac{1}{2.5.12} - \frac{1}{2.12.29} + \&c.$$

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$$+\frac{1}{4}-\frac{1}{2.\,2.\,5}+\frac{1}{2.\,5.\,12}-\frac{1}{2.\,12.\,29}+\&c.,$$

erit

$$x=\frac{1}{4}+\frac{1}{1.\,5}-\frac{1}{2.\,12}+\frac{1}{5.\,29}-\frac{1}{12.\,70}\ \&c.,$$

quæ Series etiamſi vehementer convergant, tamen vera earum ſumma ex earum forma colligi nequit.

376. Pro hujuſmodi autem fractionibus continuis, in quibus denominatores omnes vel ſunt æquales, vel iidem revertuntur; ita ut ea fractio, ſi ab initio aliquot terminis truncetur, toti adhuc ſit æqualis, facilis habetur modus earum ſummas explorandi. In exemplo enim propoſito, cum ſit

$$x=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\&c.}}}},$$

erit $x=\frac{1}{2+x}$, ideoque $xx+2x=1$ & $x+1=\sqrt{2}$; ita ut valor hujus fractionis continuæ ſit $=\sqrt{2}-1$. Fractiones vero ex fractione continua ante erutæ, continuo propius ad hunc valorem accedunt, idque tam cito, ut vix promptior modus ad valorem hunc irrationalem per numeros rationales proxime exprimendum, inveniri queat. Eſt enim $\sqrt{2}-1$ tam prope $=\frac{29}{70}$, ut error ſit inſenſibilis: namque, radicem extrahendo, erit

$$\sqrt{2}-1=0,41421356236,$$

atque

$$\frac{29}{70}=0,41428571428,$$

ita ut error tantum in partibus centeſimis milleſimis conſiſtat.

377. Quemadmodum ergo fractiones continuæ commodiſſimum ſuppeditant modum ad valorem $\sqrt{2}$ appropinquandi, ita indidem

indidem facillima via aperitur ad radices aliorum numerorum proxime inveſtigandas. Ponamus hunc in finem

$$x = \frac{1}{a + \frac{1}{a + \frac{1}{a + \frac{1}{a + \frac{1}{a + \&c.}}}}},$$

erit $x = \frac{1}{a + x}$ & $xx + ax = 1$, unde fit $x = -\frac{1}{2}a + \surd(1 + \frac{1}{4}aa) = \frac{\surd(aa+4) - a}{2}$. Hæc ergo fractio continua inſerviet valori radicis quadratæ ex numero $aa + 4$ inveniendo. Hincque adeo ſubſtituendo loco a ſucceſſive numeros 1, 2, 3, 4, &c., reperientur $\surd 5$; $\surd 2$; $\surd 13$; $\surd 5$; $\surd 29$; $\surd 10$; $\surd 53$; &c., perductis ſcilicet his radicibus ad formam ſimpliciſſimam: erit ergo

$$\overset{1}{\frac{0}{1}}, \overset{1}{\frac{1}{1}}, \overset{1}{\frac{1}{2}}, \overset{1}{\frac{2}{3}}, \overset{1}{\frac{3}{5}}, \overset{1}{\frac{5}{8}}, \&c. = \frac{\surd 5 - 1}{2}$$

$$\overset{2}{\frac{0}{1}}, \overset{2}{\frac{1}{2}}, \overset{2}{\frac{2}{5}}, \overset{2}{\frac{5}{12}}, \overset{2}{\frac{12}{29}}, \overset{2}{\frac{29}{70}}, \&c. = \surd 2 - 1$$

$$\overset{3}{\frac{0}{1}}, \overset{3}{\frac{1}{3}}, \overset{3}{\frac{3}{10}}, \overset{3}{\frac{10}{33}}, \overset{3}{\frac{33}{109}}, \overset{3}{\frac{109}{360}}, \&c. = \frac{\surd 13 - 3}{2}$$

$$\overset{4}{\frac{0}{1}}, \overset{4}{\frac{1}{4}}, \overset{4}{\frac{4}{17}}, \overset{4}{\frac{17}{72}}, \overset{4}{\frac{72}{305}}, \overset{4}{\frac{305}{1292}}, \&c. = \surd 5 - 2$$

&c.

notandum autem eo promptiorem eſſe approximationem, quo major fuerit numerus a: ſic in ultimo exemplo erit $\surd 5 = 2 \frac{305}{1292}$, ut error minor ſit quam $\frac{1}{1292 . 5473}$, ubi 5473 eſt dænominator ſequentis fractionis $\frac{1292}{5473}$.

378. Hoc vero modo aliorum numerorum radices exhiberi nequeunt, niſi qui ſint ſumma duorum quadratorum. Ut igitur hæc approximatio ad alios numeros extendatur, ponamus eſſe

$$x = \frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\&c.;$$

erit

$$x = \frac{1}{a + \frac{1}{b+x}} = \frac{b+x}{ab+1+ax};\ \text{ideoque}\ axx + abx = b,$$

&

$$x = -\tfrac{1}{2}b \pm \surd\left(\tfrac{1}{4}bb + \tfrac{b}{a}\right) = \frac{-ab + \surd(aabb + 4ab)}{2a}.$$ Unde jam omnium numerorum radices inveniri poterunt. Sit, verbi gratia, $a=2$, $b=7$; erit $x = \frac{-14+\surd 14.18}{4} = \frac{-7+3\surd 7}{2}$; at valorem ipſius x proxime exhibebunt ſequentes fractiones

$$\begin{matrix} 2, & 7, & 2, & 7, & 2, & 7 \\ \frac{0}{1}, & \frac{1}{2}, & \frac{7}{15}, & \frac{15}{32}, & \frac{112}{239}, & \frac{239}{510}, & \&c., \end{matrix}$$

Erit ergo proxime $\frac{-7+3\surd 7}{2} = \frac{239}{510}$ & $\surd 7 = \frac{2024}{765} =$ $2,6457516$; at revera eſt $\surd 7 = 2,64575131$; ita ut error minor ſit quam $\frac{3}{10000000}$.

379. Progrediamur autem ulterius ponendo

$$x = \frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{a+}\,\&c.,$$

erit

$$x = \frac{1}{a + \frac{1}{b + \frac{1}{c+x}}} = \frac{1}{a + \frac{c+x}{bx+bc+1}} = \frac{bx+bc+1}{(ab+1)x+abc+a+c};$$

unde $(ab+1)xx + (abc+a-b+c)x = bc+1$ atque

$$x = \frac{-abc - a + b - c + \sqrt{((abc + a + b + c)^2 + 4)}}{2(ab + 1)};$$ CAP. XVIII.

ubi quantitas poſt ſignum radicale poſita iterum eſt ſumma duorum quadratorum, neque ergo hæc forma radicibus ex aliis numeris extrahendis inſervit, niſi ad quos prima forma jam ſuffecerat. Simili modo ſi quatuor litteræ a, b, c, d, continuo repetitæ denominatores fractionis continuæ conſtituant, tum ea plus non inſerviet quam ſecunda, quæ duas tantum litteras continebat, & ita porro.

380. Cum igitur fractiones continuæ tam utiliter ad extractionem radicis quadratæ adhiberi queant, ſimul inſervient æquationibus quadratis reſolvendis; quod quidem ex ipſo calculo eſt manifeſtum, dum x per æquationem quadraticam affectam determinatur. Poteſt autem viciſſim facile cujuſque æquationis quadratæ radix per fractionem continuam hoc modo exprimi. Sit propoſita iſta æquatio

$$xx = ax + b,$$

ex qua, cum ſit $x = a + \frac{b}{x}$, ſubſtituatur in ultimo termino loco x valor idem jam inventus, eritque

$$x = a + \cfrac{b}{a + \cfrac{b}{x}},$$

ſimili ergo modo procedendo, erit per fractionem continuam infinitam

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \text{\&c.}}}},$$

quæ autem, cum numeratores b non ſint unitates, non tam commode adhiberi poteſt.

381. Ut autem uſus in arithmetica oſtendatur, primum notandum eſt omnem fractionem ordinariam in fractionem con-

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LIB. I. tinuam converti poſſe. Sit enim propoſita fractio $x = \frac{A}{B}$, in qua ſit $A > B$; dividatur A per B, ſitque quotus $= a$ & reſiduum C; tum per hoc reſiduum C dividatur præcedens diviſor B, prodeatque quotus b & relinquatur reſiduum D, per quod denuo præcedens diviſor C dividatur; ſicque hæc operatio, quæ vulgo ad maximum communem diviſorem numerorum A & B inveſtigandum uſurpari ſolet, continuetur, donec ipſa finiatur; ſequenti modo

$$\begin{array}{l} B\,)\,A\,(\,a \\ \quad C\,)\,B\,(\,b \\ \qquad D\,)\,C\,(\,c \\ \qquad\quad E\,)\,D\,(\,d \\ \qquad\qquad F \text{ \&c.,} \end{array}$$

eritque per naturam diviſionis

$$\begin{array}{lll} A = aB + C; \text{ unde} & \frac{A}{B} = a + \frac{C}{B}; & \\ B = bC + D; & \frac{B}{C} = b + \frac{D}{C}; & \frac{C}{B} = \frac{1}{b + \frac{D}{C}} \\ C = cD + E; & \frac{C}{D} = c + \frac{E}{D}; & \frac{D}{C} = \frac{1}{c + \frac{E}{D}} \\ D = dE + F; & \frac{D}{E} = d + \frac{F}{E}; & \frac{E}{D} = \frac{1}{d + \frac{F}{E}} \\ \text{\&c.} & \text{\&c.} & \text{\&c.} \end{array}$$

hinc, ſequentes valores in præcedentibus ſubſtituendo, erit

$$x = \frac{A}{B} = a + \frac{C}{B} = a + \frac{1}{b + \frac{D}{C}} = a + \frac{1}{b + \frac{1}{c + \frac{E}{D}}},$$

unde tandem x per meros quotos inventos a, b, c, d, &c., ſequentem in modum exprimetur, ut ſit

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \&c.}}}}}$$

EXEMPLUM I.

Sit propoſita iſta fractio $\frac{1461}{59}$, quæ ſequenti modo in fractionem continuam tranſmutabitur, cujus omnes numeratores erunt unitates. Inſtituatur ſcilicet eadem operatio, qua maximus communis diviſor numerorum 59 & 1461 quæri ſolet.

```
59) 1461 (24
    118
    ---
     281
     236
     ---
      45) 59 (1
          45
          --
          14) 45 (3
              42
              --
               3) 14 (4
                  12
                  --
                   2) 3 (1
                      2
                      -
                      1) 2 (2
                         2
                         -
                         0
```

Hinc ergo ex quotis fiet

$$\frac{1461}{59} = 24 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}}}.$$

EXEMPLUM II.

Fractiones quoque decimales eodem modo tranſmutari poterunt, ſit enim propoſita

$$\sqrt{2} = 1,41421356 = \frac{141421356}{100000000},$$

unde hæc operatio inſtituatur

100000000	141421356	1
82842712	100000000	2
17157288	41421356	2
14213560	34314576	2
2943728	7106780	2
2438648	5887456	2
505080	1219324	2
418328	1010160	2
&c.	209164	

Ex qua operatione jam patet omnes denominatores eſſe 2, atque adeo eſſe $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\&c.}}}}}$, cujus expreſſionis ratio jam ex ſuperioribus patet.

EXEMPLUM III.

Imprimis vero etiam hic attentione dignus eſt numerus e; cujus logarithmus eſt $= 1$, qui eſt $e = 2,718281828459$, unde oritur $\frac{e-1}{2} = 0,8591409142295$, quæ fractio decimalis, ſi ſuperiori modo tractetur, dabit quotos ſequentes

8591409142295	10000000000000	1
8451545146224	8591409142295	6
139863996071	1408590857704	10
139312557916	1398639960710	14
551438155	9950896994	18
550224488	9925886790	22
1213667	25010204	
	&c.	

ſi iſte calculus exactius adhuc, aſſumto valore ipſius e, ulterius continuetur, tum prodibunt iſti quoti

1, 6, 10, 14, 18, 22, 26, 30, 34, &c.,

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qui, demto primo, progreſſionem arithmeticam conſtituunt, CAP. XVIII.
unde patet fore

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\&c.}}}}}},$$

cujus fractionis ratio ex calculo infiniteſimali dari poteſt.

382. Cum igitur ex hujuſmodi expreſſionibus fractiones erui queant, quæ quam citiſſime ad verum valorem expreſſionis deducant, hæc methodus adhiberi poterit ad fractiones decimales per ordinarias fractiones, quæ ad ipſas proxime accedant, exprimendas. Quin etiam, ſi fractio fuerit propoſita cujus numerator & denominator ſint numeri valde magni, fractiones ex minoribus numeris conſtantes inveniri poterunt quæ, etiamſi propoſitæ non ſint penitus æquales, tamen ab ea quam minime diſcrepent. Hincque problema a Wallisio olim tractatum facile reſolvi poteſt, quo quæruntur fractiones minoribus numeris expreſſæ, quæ tam prope exhauriant valorem fractionis cujuſpiam in numeris majoribus propoſitæ, quantum fieri poterit numeris non majoribus. Fractiones autem noſtra hac methodo ortæ tam prope ad valorem fractionis continuæ, ex qua eliciuntur, accedunt, ut nullæ numeris non majoribus conſtantes dentur quæ propius accedant.

Exemplum I.

Exprimatur ratio diametri ad peripheriam numeris tam exiguis, ut accuratior exhiberi nequeat, niſi numeri majores adhibeantur. Si fractio decimalis cognita

3, 1415926535 &c.,

modo expoſito per diviſionem continuam evolvatur, reperientur ſequentes quoti

3, 7, 15, 1, 292, 1, 1, &c.,

ex quibus ſequentes fractiones formabuntur,

$\frac{1}{0}, \frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}$, &c.,

 ſecunda fractio jam oſtendit eſſe diametrum ad peripheriam ut 1 : 3, neque certe numeris non majoribus accuratius dari poterit. Tertia fractio dat rationem *Archimedeam* 7 : 22, at quinta *Metianam* quæ ad verum tam prope accedit, ut error minor ſit parte $\frac{1}{113 . 33102}$. Ceterum hæ fractiones alternatim vero ſunt majores minoreſque.

Exemplum II.

Exprimatur ratio diei ad annum ſolarem medium in numeris minimis proxime. Cum annus iſte ſit $365^d, 5^h, 48', 55''$, continebit in fractione annus unus $365 \frac{20935}{86400}$ dies. Tantum ergo opus eſt ut hæc fractio evolvatur, quæ dabit ſequentes quotos

$$4, 7, 1, 6, 1, 2, 2, 4$$

unde iſtæ eliciuntur fractiones

$$\frac{0}{1}, \frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{55}{227}, \frac{63}{260}, \frac{181}{747}, \text{\&c.}$$

Horæ ergo cum minutis primis & ſecundis, quæ ſupra 365 dies adſunt, quatuor annis unum diem circiter faciunt, unde calendarium *Julianum* originem habet. Exactius autem 33 annis 8 dies implentur, vel 747 annis 181 dies; unde ſequitur quadringentis annis abundare 97 dies. Quare, cum hoc intervallo calendarium *Julianum* inferat 100 dies, *Gregorianus* quaternis ſeculis tres annos biſſextiles in communes convertit.

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